Chapter 18

Vector Analysis

The fundamental theorem of calculus is extended from the line to the plane and to space.

The major theorems of vector analysis relate double and triple integrals to integrals over curves and surfaces. These results have their origins in problems of fluid flow and electromagnetic theory; thus, it is not surprising that they are very important for physics as well as for their own mathematical beauty. Enough applications are given in this chapter so that the student can appreciate the physical meaning of the theorems.

18.1 Line Integrals

A vector field may be integrated along a curve to produce a number.

The integration of vector fields along curves is of fundamental importance in both mathematics and physics. We will use the concept of *work* to motivate the material in this section. In later sections, we establish Green's and Stokes' theorems, which relate line integrals, partial derivatives, and double integrals.

The motion of an object is described by a parametric curve—that is, by a vector function $\mathbf{r} = \boldsymbol{\sigma}(t)$. By differentiating this function, we obtain (see Section 14.6):

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\sigma}'(t) = \text{velocity at time } t;$$

$$v = \|\mathbf{v}\| = \|\boldsymbol{\sigma}'(t)\| = \text{speed at time } t;$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \boldsymbol{\sigma}''(t) = \text{acceleration at time } t.$$

According to Newton's second law,

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{r}}{dt^2} = m\boldsymbol{\sigma}''(\mathbf{t}),$$

where $\mathbf{F}$ is the total force acting on the object. If the mass of the object is m, the kinetic energy K is defined by

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v}.$$

To investigate the relationship between force and kinetic energy, we differentiate K with respect to t, obtaining

$$\frac{dK}{dt} = \frac{1}{2}m\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}.$$

The total change in kinetic energy from time t_1 to t_2 is the integral of dK/dt, so we get

$$\Delta K = \int_{t_1}^{t_2}\frac{dK}{dt}\,dt = \int_{t_1}^{t_2}\mathbf{F}\cdot\mathbf{v}\,dt = \int_{t_1}^{t_2}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

The integral

$$\int_{t_1}^{t_2}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt, \tag{1}$$

denoted W, is called the *work* done by the force $\mathbf{F}$ along the path $\mathbf{r} = \boldsymbol{\sigma}(t)$.

It is often possible to express the total force on an object as a sum of forces which are due to identifiable sources (such as gravity, friction, and fluid pressure). If $\mathbf{F}$ represents a force of a particular type, then the integral (1) is still called the work done by this particular force.

Let us now suppose that the force $\mathbf{F}$ at time t depends only on the position $\mathbf{r} = \boldsymbol{\sigma}(t)$. That is, we assume that there is a *vector field* $\boldsymbol{\Phi}(\mathbf{r})$ such that $\mathbf{F} = \boldsymbol{\Phi}(\boldsymbol{\sigma}(t))$. (Examples of such *position-dependent* forces are those caused by gravitational and electrostatic attraction; frictional and magnetic forces are *velocity dependent.*) Then we may write the integral (1) as

$$W = \int_{t_1}^{t_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}(t))\cdot\boldsymbol{\sigma}'(t)\,dt. \tag{2}$$

In the one-dimensional case, (2) can be simplified, by a change of variables from time to position, to

$$W = \int_a^b F(x)\,dx,$$

where a and b are the starting and ending positions. This formula agrees with that found in Section 9.5. Notice that the work done depends only on F, a, and b and not on the details of the motion. We shall prove later in this section that, to a certain extent, the same situation remains true for motion in space: the work done by a force field as a particle moves along a path does not depend upon how the particle moves along the path; however, if *different paths* are taken between the same endpoints, the work may be different.

Example 1 Find the work done by the force field $\boldsymbol{\Phi}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ as a particle is moved from $(1,0,0)$ to $(1,0,1)$ along each of the following paths:

(a) $(x, y, z) = (\cos t, \sin t, t/2\pi)$; $0 \leqslant t \leqslant 2\pi$;
(b) $(x, y, z) = (\cos t^3, \sin t^3, t^3/2\pi)$; $0 \leqslant t \leqslant \sqrt[3]{2\pi}$;
(c) $(x, y, z) = (\cos t, -\sin t, t/2\pi)$; $0 \leqslant t \leqslant 2\pi$.

Solution The (helical) paths are sketched in Fig. 18.1.1.

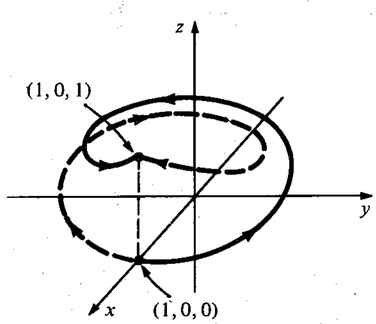

Figure 18.1.1. Paths (a) and (b) follow the solid line; path (c) follows the dashed line.

(a) By formula (2), with $t_1 = 0$, $t_2 = 2\pi$, $\boldsymbol{\sigma}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (t/2\pi)\mathbf{k}$, and $\boldsymbol{\sigma}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + (1/2\pi)\mathbf{k}$, the work done by the force along path (a) is

$$\begin{aligned} W_a &= \int_0^{2\pi} (\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}) \cdot \left(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \frac{1}{2\pi}\mathbf{k} \right) dt \\ &= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + \frac{1}{2\pi} \right) dt \\ &= 2\pi\left(-1 + \frac{1}{2\pi} \right) = -2\pi + 1 \approx -5.28. \end{aligned}$$

(b) This time,

$$\begin{aligned} \boldsymbol{\sigma}'(t) &= -(\sin t^3)(3t^2)\mathbf{i} + (\cos t^3)(3t^2)\mathbf{j} + \frac{3t^2}{2\pi}\mathbf{k} \\ &= 3t^2\left[-\sin t^3\,\mathbf{i} + \cos t^3\,\mathbf{j} + \frac{1}{2\pi}\mathbf{k} \right], \end{aligned}$$

and

$$\begin{aligned} W_b &= \int_0^{\sqrt[3]{2\pi}} (\sin t^3\,\mathbf{i} - \cos t^3\,\mathbf{j} + \mathbf{k}) \cdot (3t^2)\left(-\sin t^3\,\mathbf{i} + \cos t^3\,\mathbf{j} + \frac{1}{2\pi}\mathbf{k} \right) dt \\ &= \int_0^{\sqrt[3]{2\pi}} \left(-\sin^2 t^3 - \cos^2 t^3 + \frac{1}{2\pi} \right) 3t^2\,dt \\ &= \int_0^{\sqrt[3]{2\pi}} \left(-1 + \frac{1}{2\pi} \right) 3t^2\,dt = \left(-1 + \frac{1}{2\pi} \right) t^3 \Big|_{t=0}^{\sqrt[3]{2\pi}} \\ &= \left(-1 + \frac{1}{2\pi} \right)(2\pi) = 1 - 2\pi, \end{aligned}$$

just as in part (a).
(c) Here

$$\boldsymbol{\sigma}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \frac{1}{2\pi}\mathbf{k},$$

so

$$\begin{aligned} W_c &= \int_0^{2\pi} (-\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}) \cdot \left(-\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \frac{1}{2\pi}\mathbf{k} \right) dt \\ &= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t + \frac{1}{2\pi} \right) dt \\ &= 2\pi\left(1 + \frac{1}{2\pi} \right) = 2\pi + 1 \approx 7.28. \end{aligned}$$

In the case of path (c), the motion is "with the force," so the work is positive; for paths (a) and (b), the motion is "against the force" and the work is negative. ▲

The equality of work along paths (a) and (b) in Example 1 is no accident: it is a consequence of the fact that the two paths are simply two different parametrizations of the same curve in space. Shortly, we will prove that the work along a path is always independent of the parametrization, and in Section 18.2, we will identify those special force fields for which the work is independent of the path itself.

Example 2 Suppose that in the force field of Example 1, you pick up a unit mass at $(1,0,0)$, carry it along path (a), and leave it at $(1,0,1)$. How much work have *you* done?

Solution Since the kinetic energy is zero at the beginning and the end of the process, the change in energy is zero and the total work is zero. The two sources of work are you and the force field; since the work done by force field is -5.28, by part (a) of Example 1, the work done by you must be 5.28. ▲

Example 3 Consider the gravitational force field defined (for $(x,y,z)\neq(0,0,0)$) by

$$\mathbf{\Phi}(x,y,z)=\frac{-1}{\left(x^2+y^2+z^2\right)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

Show that the work done by the gravitational force as a particle moves from (x_1,y_1,z_1) to (x_2,y_2,z_2) depends only on the radii $R_1=\sqrt{x_1^2+y_1^2+z_1^2}$ and $R_2=\sqrt{x_2^2+y_2^2+z_2^2}$.

Solution Let the path be given by $(x,y,z)=\boldsymbol{\sigma}(t)$, where $\boldsymbol{\sigma}(t_1)=(x_1,y_1,z_1)$ and $\boldsymbol{\sigma}(t_2)=(x_2,y_2,z_2)$. Think of x, y, and z as functions of t. Then $\boldsymbol{\sigma}'(t)=(dx/dt)\mathbf{i}+(dy/dt)\mathbf{j}+(dz/dt)\mathbf{k}$, and so

$$\begin{aligned}
W&=\int_{t_1}^{t_2}\frac{-1}{\left(x^2+y^2+z^2\right)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\left(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}\right)dt\\
&=-\int_{t_1}^{t_2}\frac{x\,(dx/dt)+y\,(dy/dt)+z\,(dz/dt)}{\left(x^2+y^2+z^2\right)^{3/2}}\,dt\\
&=\int_{t_1}^{t_2}-\frac{(1/2)(d/dt)(x^2+y^2+z^2)}{\left(x^2+y^2+z^2\right)^{3/2}}\,dt\\
&=\int_{t_1}^{t_2}\frac{d}{dt}\left(x^2+y^2+z^2\right)^{-1/2}dt=\left(x^2+y^2+z^2\right)^{-1/2}\Big|_{\boldsymbol{\sigma}(t_1)}^{\boldsymbol{\sigma}(t_2)}\\
&=\left(x_2^2+y_2^2+z_2^2\right)^{-1/2}-\left(x_1^2+y_1^2+z_1^2\right)^{-1/2}=\frac{1}{R_2}-\frac{1}{R_1}.
\end{aligned}$$

Thus, the work done by the gravitational field when a particle moves from (x_1,y_1,z_1) to (x_2,y_2,z_2) is $1/R_2-1/R_1$. Notice that, in this case, the work is independent of the path taken between the two points. ▲

To define the line integral, we use formula (2) abstracted from its physical interpretation.

The Line Integral

Let $\mathbf{\Phi}$ be a vector field defined in some region of space, and let $\mathbf{r}=\boldsymbol{\sigma}(t)$ be a parametric curve in that region defined for t in $[t_1,t_2]$. The integral

$$\int_{t_1}^{t_2}\mathbf{\Phi}(\boldsymbol{\sigma}(t))\cdot\boldsymbol{\sigma}'(t)\,dt \tag{3}$$

is called the *line integral* of the vector field $\mathbf{\Phi}$ along this curve.

The work done by a force field on a moving particle is therefore the line integral of the force along the path travelled by the particle.

Example 4 Find the line integral of the vector field $e^y\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$ along the curve $(0, t, t^2)$, for $0 \leqslant t \leqslant \ln 2$.

Solution The velocity vector $\boldsymbol{\sigma}'(t)$ is $\mathbf{j} + 2t\mathbf{k}$, so the line integral is

$$\int_0^{\ln 2}(e^t\mathbf{i} + e^0\mathbf{j} + e^{t^2}\mathbf{k}) \cdot (\mathbf{j} + 2t\mathbf{k})\,dt = \int_0^{\ln 2}(1 + e^{t^2}2t)\,dt$$
$$= (t + e^{t^2})\big|_0^{\ln 2} = \ln 2 + e^{(\ln 2)^2} - 1$$
$$= \ln 2 + 2^{\ln 2} - 1 \approx 1.31. \blacktriangle$$

The next theorem shows that the line integral does not depend upon how the path of integration is parametrized.

Theorem: Independence of Parametrization

Given $\boldsymbol{\Phi}$ and $\boldsymbol{\sigma}$ as in the preceding box, let $t = f(u)$ be a differentiable function defined on the interval $[u_1, u_2]$ such that $f(u_1) = t_1$ and $f(u_2) = t_2$. *Let* $\boldsymbol{\sigma}_1(u)$ *be the parametric curve defined by* $\boldsymbol{\sigma}_1(\mathrm{u}) = \boldsymbol{\sigma}(\mathrm{f(u)})$. *Then*

$$\int_{u_1}^{u_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}_1(u)) \cdot \boldsymbol{\sigma}_1'(u)\,du = \int_{t_1}^{t_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt.$$

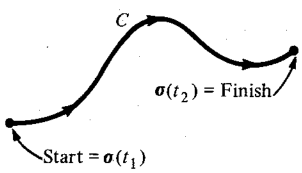

Figure 18.1.2. A geometric curve must be parametrized in a specified direction.

The basic idea of the proof was illustrated in Example 1. We apply the chain rule to $\boldsymbol{\sigma}_1(u) = \boldsymbol{\sigma}(f(u))$, obtaining $\boldsymbol{\sigma}_1'(u) = \boldsymbol{\sigma}'(f(u))f'(u)$. Hence

$$\int_{u_1}^{u_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}_1(u)) \cdot \boldsymbol{\sigma}_1'(u)\,du = \int_{u_1}^{u_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}(f(u))) \cdot \boldsymbol{\sigma}'(f(u))f'(u)\,du.$$

We next change variables from u to $t = f(u)$. Since $dt = f'(u)\,du$, the integral becomes $\int_{t_1}^{t_2}\boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt$ as required. ■

A *geometric curve* C is a set of points in the plane which can be traversed by a parametrized curve; the direction of travel along C is specified, but not a specific parametrization (see Fig. 18.1.2). The theorem shows that the line integral of a vector field along a geometric curve is well defined.

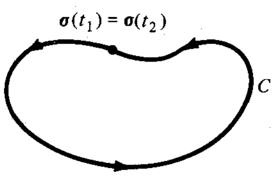

Figure 18.1.3. Any point may be taken as the starting point for integration around a closed curve.

A parametric curve $\boldsymbol{\sigma}(t)$ defined on $[t_1, t_2]$ is called *closed* if its endpoints coincide—that is, if $\boldsymbol{\sigma}(t_1) = \boldsymbol{\sigma}(t_2)$. A geometric curve is *closed* if it has a parametrization which is closed. When C is a closed curve, any point of C may be taken as the initial point for the parametrization, but we must be sure to go around C just once (see Fig. 18.1.3 and Example 5.)

In summary, there are two reservations which must be noted in choosing a parametrization of a geometric curve: *the parametrization must go in the correct direction and it must trace out the curve exactly once.*

Example 5 (a) Let C be the line segment joining $(0,0,0)$ to $(1,0,0)$ and let $\boldsymbol{\sigma}_1(t) = (t, 0, 0)$, $0 \leqslant t \leqslant 1$. Find the line integral of $\boldsymbol{\Phi}(x, y, z) = \mathbf{i}$ along this curve. If C is parametrized by $\boldsymbol{\sigma}_2(t) = (1 - t, 0, 0)$, $0 \leqslant t \leqslant 1$, find the line integral.
(b) Let C be the circle given by $x^2 + y^2 = 1$, $z = 0$. Let $\boldsymbol{\sigma}_1(t) = (\cos t, \sin t, 0)$, $0 \leqslant t \leqslant 2\pi$. Find the line integral of $\boldsymbol{\Phi}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. If C is parametrized by $\boldsymbol{\sigma}_2(t) = (\cos t, \sin t, 0)$, $0 \leqslant t \leqslant 4\pi$, find the line integral.

Solution (a) Here $\sigma_1'(t) = \mathbf{i}$, $t_1 = 0$, $t_2 = 1$, and $\mathbf{\Phi}(\boldsymbol{\sigma}_1(t)) = \mathbf{i}$, so formula (3) gives

$$\int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}_1(t)) \cdot \boldsymbol{\sigma}_1'(t)\,dt = \int_0^1 \mathbf{i} \cdot \mathbf{i}\,dt = \int_0^1 dt = 1.$$

For $\boldsymbol{\sigma}_2$, we similarly have $t_1 = 0$, $t_2 = 1$, $\boldsymbol{\sigma}_2'(t) = -\mathbf{i}$, and $\mathbf{\Phi}(\boldsymbol{\sigma}_2(t)) = \mathbf{i}$, so

$$\int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}_2(t)) \cdot \boldsymbol{\sigma}_2'(t)\,dt = \int_0^1 -\mathbf{i} \cdot \mathbf{i}\,dt = -\int_0^1 dt = -1.$$

Here the geometric curve C is the same, but the two parametrizations, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, traverse C in opposite directions. See Fig. 18.1.4.

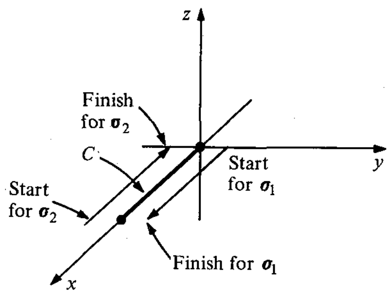

Figure 18.1.4. $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ traverse C in opposite directions.

(b) The line integral for $\boldsymbol{\sigma}_1$ is obtained from formula (3) by substituting $t_1 = 0$, $t_2 = 2\pi$, $\boldsymbol{\sigma}_1(t) = (\cos t, \sin t, 0)$, $\mathbf{\Phi}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$, $\mathbf{\Phi}(\boldsymbol{\sigma}_1(t)) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, $\boldsymbol{\sigma}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, and $\boldsymbol{\sigma}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, as follows:

$$\int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}_1(t)) \cdot \boldsymbol{\sigma}_1'(t)\,dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi.$$

Notice that we get the same result if we choose any θ and parametrize C by the equation $\boldsymbol{\sigma}(t) = (\cos(t + \theta), \sin(t + \theta), 0)$, $0 \leqslant t \leqslant 2\pi$; this will start and finish at $(\cos\theta, \sin\theta, 0)$. If we go backwards along C, using the parametrization $\boldsymbol{\sigma}(t) = (\cos(-t), \sin(-t), 0)$, we will get the *negative* of our earlier answer.

If we use $\boldsymbol{\sigma}_2(t)$, the only change is that t_2 is changed to 4π, so we get

$$\int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}_2(t)) \cdot \boldsymbol{\sigma}_2'(t)\,dt = 4\pi.$$

This is double our first answer, since $\boldsymbol{\sigma}_2$ traverses C twice. See Fig. 18.1.5. ▲

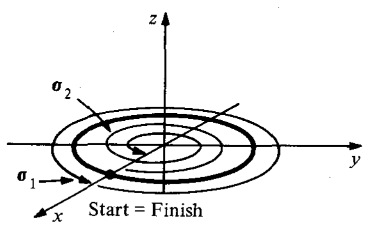

Figure 18.1.5. $\boldsymbol{\sigma}_1$ goes around C once while $\boldsymbol{\sigma}_2$ goes around twice.

A useful notation for the line integral is suggested by the Leibniz notation. Let us write $\mathbf{r} = \boldsymbol{\sigma}(t)$ so that $\boldsymbol{\sigma}'(t) = d\mathbf{r}/dt$ and (3) becomes $\int_{t_1}^{t_2} \mathbf{\Phi}(\mathbf{r}) \cdot (d\mathbf{r}/dt)\,dt$. It is tempting to change variables to $\mathbf{r}$ and write $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$, where $\mathbf{r}_1 = \boldsymbol{\sigma}(t_1)$ and $\mathbf{r}_2 = \boldsymbol{\sigma}(t_2)$, but this notation does not display the curve traced out by $\mathbf{r} = \boldsymbol{\sigma}(t)$, only its endpoints. If we use the letter C to denote the path of integration, however, we can define

$$\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt, \tag{4}$$

where $\boldsymbol{\sigma}(t)$ is any parametrization of C (subject to the reservations discussed above). The notation $d\mathbf{r}$ for $\boldsymbol{\sigma}'(t)\,dt$ is consistent with our other uses of

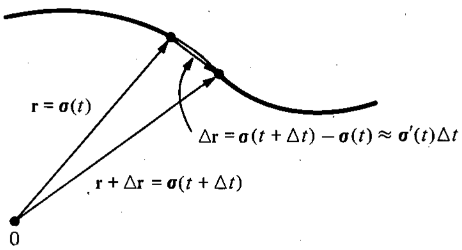

Figure 18.1.6. $\Delta\mathbf{r} \approx \boldsymbol{\sigma}'(t)\,\Delta t$.

infinitesimal notation. The change in $\mathbf{r}$ over a small time interval Δt is $\Delta\mathbf{r} \approx \boldsymbol{\sigma}'(t)\Delta t$ (see Fig. 18.1.6). As Δt becomes the infinitesimal dt, $\Delta\mathbf{r}$ passes over into $d\mathbf{r}$, and the approximate equality becomes exact.

Example 6 Let C be the straight line segment joining $(2,1,3)$ to $(-4,6,8)$. Find

$$\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r},$$

where $\boldsymbol{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$.

Solution We may choose any parametrization of C; the simplest is probably

$$\begin{aligned}\boldsymbol{\sigma}(t) &= (1-t)(2,1,3) + t(-4,6,8)\\ &= (2-6t, 1+5t, 3+5t), \qquad 0 \leqslant t \leqslant 1.\end{aligned}$$

As t varies from 0 to 1, $\boldsymbol{\sigma}(t)$ moves along C from $(2,1,3)$ to $(-4,6,8)$. By formula (4), we get

$$\begin{aligned}&\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}\\ &\quad= \int_0^1 \left[(2-6t)\mathbf{i} - (1+5t)\mathbf{j} + (2-6t)(1+5t)\mathbf{k}\right] \cdot (-6\mathbf{i} + 5\mathbf{j} + 5\mathbf{k})\,dt\\ &\quad= \int_0^1 (-7 + 31t - 150t^2)\,dt = -\frac{83}{2}. \blacktriangle\end{aligned}$$

Example 7 Suppose that $\boldsymbol{\Phi}(\mathbf{r})$ is orthogonal to $\boldsymbol{\sigma}'(t)$ at each point of the curve $\boldsymbol{\sigma}(t)$. What can you say about the line integral $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$?

Solution Since $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt$, this integral will be zero because $\boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t) = 0$, as $\boldsymbol{\Phi}$ and $\boldsymbol{\sigma}'$ are orthogonal. ▲

The formulas

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \tag{5}$$

and

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{6}$$

for ordinary integrals have their counterparts in line integration. If we choose a point on a curve C, it divides C into two curves, C_1 and C_2 (see Fig. 18.1.7(a)). We write $C = C_1 + C_2$. Then (4) and (5) give

$$\int_{C_1 + C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}. \tag{7}$$

Let $-C$ be the curve C traversed in the reverse direction (see Fig. 18.1.7(b)). Then (4) and (6) give

$$\int_{-C} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = -\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}. \tag{8}$$

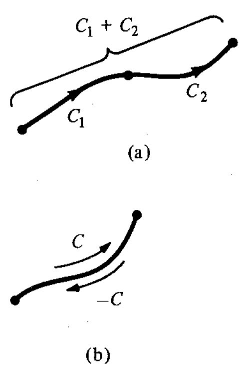

Figure 18.1.7. "Algebraic" operations on curves.

Addition formula (7) suggests a way to define line integrals over curves with "corners"—that is, continuous curves on which the tangent vector is undefined at certain points. If C is such a curve, we write $C = C_1 + C_2 + C_3 + \cdots + C_n$ by dividing it at the corner points, and we *define* $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ to be

$$\sum_{i=1}^{n} \int_{C_i} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} \qquad \text{(see Fig. 18.1.8).}$$

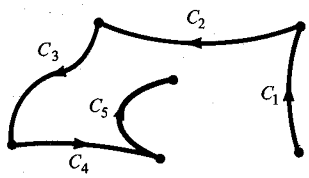

Figure 18.1.8. If $C = C_1 + \cdots + C_n$, then $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \sum_{i=1}^{n} \int_{C_i} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$.

Example 8 Let C be the perimeter of the unit square $[0,1] \times [0,1]$ in the plane, traversed in the counterclockwise direction (see Fig. 18.1.9). Evaluate the line integral $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$, where $\mathbf{\Phi}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$.

Solution We do this problem by integrating along each of the sides C_1, C_2, C_3, C_4 separately and adding the results (see Fig. 18.1.9). The parametrizations are

C_1: $(t, 0)$, $0 \leqslant t \leqslant 1$; $\boldsymbol{\sigma}_1(t) = t\mathbf{i}$.
C_2: $(1, t)$, $0 \leqslant t \leqslant 1$; $\boldsymbol{\sigma}_2(t) = \mathbf{i} + t\mathbf{j}$.
C_3: $(1-t, 1)$, $0 \leqslant t \leqslant 1$; $\boldsymbol{\sigma}_3(t) = (1-t)\mathbf{i} + \mathbf{j}$.
C_4: $(0, 1-t)$, $0 \leqslant t \leqslant 1$; $\boldsymbol{\sigma}_4(t) = (1-t)\mathbf{j}$.

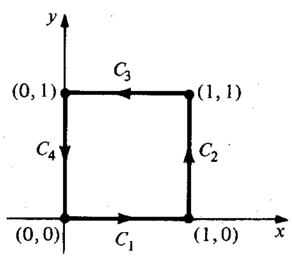

Figure 18.1.9. The perimeter of the unit square broken into four pieces.

Thus, by (4),

$$\int_{C_1} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 t^2\, dt = \frac{1}{3},$$

$$\int_{C_2} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{1}{2},$$

$$\int_{C_3} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 (1-t)^2(-1)\, dt = -\frac{1}{3},$$

$$\int_{C_4} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 0\, dt = 0.$$

Adding, we get

$$\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{2}. \; \blacktriangle$$

Another notation for the line integral arises if we write our vectors in components. Suppose that $\mathbf{\Phi}(x, y, z) = a(x, y, z)\mathbf{i} + b(x, y, z)\mathbf{j} + c(x, y, z)\mathbf{k}$.

The expression for the derivative,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

can be written formally as

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k},$$

and so

$$\begin{aligned} \int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} &= \int_C \left[a(x,y,z)\mathbf{i} + b(x,y,z)\mathbf{j} + c(x,y,z)\mathbf{k} \right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \\ &= \int_C \left[a(x,y,z)\,dx + b(x,y,z)\,dy + c(x,y,z)\,dz \right]. \end{aligned} \tag{9}$$

The expression inside the last integral is called a *differential form*. To evaluate the integral, we choose a parametrization of C. Then x, y, and z become functions of t; dx, dy, dz are expressed in terms of t and dt, and the integrand becomes an ordinary integrand in t which may be integrated over the parametrization interval.

Example 9 Evaluate $\int_C \cos z\,dx + e^x\,dy + e^y\,dz$, where C is parametrized by $(x, y, z) = (1, t, e^t)$, $0 \leqslant t \leqslant 2$.

Solution We compute $dx = 0$, $dy = dt$, $dz = e^t dt$, and so

$$\int_C [\cos z\,dx + e^x\,dy + e^y\,dz] = \int_0^2 (0 + e^1 + e^{2t})\,dt$$

$$= \left[et + \frac{1}{2}e^{2t} \right]_0^2 = 2e + \frac{1}{2}e^4 - \frac{1}{2}. \blacktriangle$$

The following box summarizes the notations and definitions developed so far.

Line Integrals

The line integral of a vector function (x, y, z) along a curve $\boldsymbol{\sigma}(t)$, $t_1 \leqslant t \leqslant t_2$, is

$$\int_{t_1}^{t_2} \boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt.$$

If $\boldsymbol{\Phi}(x, y, z) = a(x, y, z)\mathbf{i} + b(x, y, z)\mathbf{j} + c(x, y, z)\mathbf{k}$, and $\boldsymbol{\sigma}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\int_{t_1}^{t_2} \boldsymbol{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\,dt = \int_{t_1}^{t_2} a(x(t), y(t), z(t))\,\frac{dx}{dt}\,dt$$

$$+ \int_{t_1}^{t_2} b(x(t), y(t), z(t))\,\frac{dy}{dt}\,dt + \int_{t_1}^{t_2} c(x(t), y(t), z(t))\,\frac{dz}{dt}\,dt,$$

which is also written

$$\int_C a\,dx + b\,dy + c\,dz \quad \text{or} \quad \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} \quad \text{or simply} \quad \int_C \boldsymbol{\Phi}.$$

Example 10 Find $\int_C \sin \pi x\,dy - \cos \pi y\,dz$, where C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in that order.

Solution We write $C = C_1 + C_2 + C_3$, where C_1 is the line segment from $(1,0,0)$ to $(0,1,0)$, C_2 is the line segment from $(0,1,0)$ to $(0,0,1)$, and C_3 is the line segment from $(0,0,1)$ to $(1,0,0)$. Parametrizations on $[0,1]$ for these segments are

C_1: $(x, y, z) = (1 - t, t, 0)$ so $dx = -dt$, $dy = dt$, $dz = 0$;
C_2: $(x, y, z) = (0, 1 - t, t)$ so $dx = 0$, $dy = -dt$, $dz = dt$;
C_3: $(x, y, z) = (t, 0, 1 - t)$ so $dx = dt$, $dy = 0$, $dz = -dt$.

Then

$$\begin{aligned}
\int_C \sin(\pi x)\,dy - \cos(\pi y)\,dz &= \sum_{i=1}^{3} \int_{C_i} \sin(\pi x)\,dy - \cos(\pi y)\,dz \\
&= \int_0^1 \sin[\pi(1-t)]\,dt - \cos(\pi t)\cdot 0 \\
&\quad + \int_0^1 \sin(\pi\cdot 0)(-dt) - \cos[\pi(1-t)]\,dt \\
&\quad + \int_0^1 \sin\pi t\cdot 0 - \cos(\pi\cdot 0)(-dt) \\
&= \int_0^1 \sin[\pi(1-t)]\,dt - \int_0^1 \cos[\pi(1-t)]\,dt + \int_0^1 dt \\
&= -\frac{1}{\pi}\{-\cos[\pi(1-t)]\}\Big|_0^1 - \left(-\frac{1}{\pi}\right)\{\sin[\pi(1-t)]\}\Big|_0^1 + 1 \\
&= \frac{1}{\pi}(\cos 0 - \cos\pi) + \frac{1}{\pi}(\sin 0 - \sin\pi) + 1 \\
&= \frac{1}{\pi}[1-(-1)] + \frac{1}{\pi}(0-0) + 1 = \frac{2}{\pi} + 1. \blacktriangle
\end{aligned}$$

Exercises for Section 18.1

1. Calculate the work which is done by the force field $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ when a particle is moved along the path $(3t^2, t, 1)$, $0 \leqslant t \leqslant 1$.
2. Find the work done by the force field in Exercise 1 when a particle is moved along the straight line segment from $(0,0,1)$ to $(3,1,1)$.
3. Find the work which is done by the force field $\mathbf{\Phi}(x, y) = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $(x, y) = (\cos t, \sin t)$, $0 \leqslant t \leqslant 2\pi$.
4. Find the work done by the force field in Exercise 3 around the loop $(x, y) = (1 + \cos t, 1 + \sin t)$, $0 \leqslant t \leqslant 2\pi$.
5. Suppose that you pick up a unit mass which was at rest at $(1,0,0)$ and carry it to $(1,0,1)$ along the path $(1,0,t)$ under the field $xy\mathbf{i} + (x+y)\mathbf{k}$. If you leave the particle with velocity vector $\mathbf{i} + 2\mathbf{j}$ at the end of the trip, how much work have you done?
6. Do as in Exercise 5, except that the particle is left at rest at the end of the trip.
7. Show that if a particle is moved along the closed curve $(\cos t, \sin t, 0)$, $0 \leqslant t \leqslant 2\pi$, then the force field in Example 1 does a nonzero amount of work on the particle. How much is the work?
8. Show that if a particle is moved along a closed curve (that is, $\boldsymbol{\sigma}(t_1) = \boldsymbol{\sigma}(t_2)$), then the work done on it by the gravitational field in Example 3 is zero.

Let $\mathbf{\Phi}(x, y) = [1/(x^2 + y^2)](-y\mathbf{i} + x\mathbf{j})$ be a force field in the plane (minus the origin). Compute the work done by this force along each of the paths in Exercises 9–12.

9. $(\cos t, \sin t)$; $0 \leqslant t \leqslant \pi$
10. $(\cos t, -\sin t)$; $0 \leqslant t \leqslant \pi$
11. $(\cos t, \sin t)$; $0 \leqslant t \leqslant 2\pi$
12. $(-\cos t, \sin t)$; $0 \leqslant t \leqslant 2\pi$

In Exercises 13–20, evaluate the integral of the given vector field $\mathbf{\Phi}$ along the given path.

13. $\boldsymbol{\sigma}(t) = (\sin t, \cos t, t)$, $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
14. $\boldsymbol{\sigma}(t) = (t, t, t)$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$.
15. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, 0)$; $0 \leqslant t \leqslant \pi/2$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$.
16. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, 0)$; $0 \leqslant t \leqslant \pi/2$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + 2\mathbf{k}$.
17. $\boldsymbol{\sigma}(t) = (\sin t, t^2, t)$; $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = \sin z\mathbf{i} + \cos\sqrt{y}\,\mathbf{j} + x^3\mathbf{k}$.
18. $\boldsymbol{\sigma}(t) = (\cos t, \sec t, \tan t)$; $-\pi/4 \leqslant t \leqslant \pi/4$, $\mathbf{\Phi}(x, y, z) = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$.
19. $\boldsymbol{\sigma}(t) = ((1 + t^2)^2, 1, t)$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = [1/(z^2 + 1)]\mathbf{i} + x(1 + y^2)\mathbf{j} + e^y\mathbf{k}$.
20. $\boldsymbol{\sigma}(t) = 3t\mathbf{i} + (t-1)\mathbf{j} + t^2\mathbf{k}$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = (x^2 + x)\mathbf{i} + \dfrac{x-y}{x+y}\mathbf{j} + (z - z^3)\mathbf{k}$.

Let $\mathbf{\Phi}(x, y, z) = x^2\mathbf{i} - xy\mathbf{j} + \mathbf{k}$. Evaluate the line integral of $\mathbf{\Phi}$ along each of the curves in Exercises 21–24.

21. The straight line joining $(0,0,0)$ to $(1,1,1)$.
22. The circle of radius 1, center at the origin and lying in the yz plane, traversed counterclockwise as viewed from the positive x axis.
23. The parabola $z = x^2$, $y = 0$, between $(-1,0,1)$ and $(1,0,1)$.
24. The straight line between $(-1,0,1)$ and $(1,0,1)$.
25. Let C be parametrized by $x = \cos^3\theta$, $y = \sin^3\theta$, $z = \theta$, $0 \leqslant \theta \leqslant 7\pi/2$. Evaluate the integral $\int_C \sin z\,dx + \cos z\,dy - (xy)^{1/3}\,dz$.

26. Evaluate $\int_C x^2\,dx + xy\,dy + dz$, where C is parametrized by $\boldsymbol{\sigma}(t) = (t, t^2, 1)$, $0 \leqslant t \leqslant 1$.
27. Evaluate $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$, where $\boldsymbol{\Phi}(x, y, z) = \sin z\mathbf{i} + \cos\sqrt{y}\,\mathbf{j} + x^3\mathbf{k}$ and C is the line segment from $(1,0,0)$ to $(0,0,3)$.
28. Evaluate $\int_C e^{x+y-z}(\mathbf{i}+\mathbf{j}-\mathbf{k}) \cdot d\mathbf{r}$, where C is the path $(\ln t, t, t)$ for $2 \leqslant t \leqslant 4$.

The *line integral of a scalar function* f along a parametric curve $\boldsymbol{\sigma}(t)$, $t_1 \leqslant t \leqslant t_2$, is defined by

$$\int_{t_1}^{t_2} f(\boldsymbol{\sigma}(t))\|\boldsymbol{\sigma}'(t)\|\,dt.$$

Note that if $f = 1$, this is just the arc length of the curve. Evaluate the line integrals of the functions along the indicated curves in Exercises 29–34.

29. $f(x, y, z) = x + y + yz$, where $\boldsymbol{\sigma}(t) = (\sin t, \cos t, t)$; $0 \leqslant t \leqslant 2\pi$.
30. $f(x, y, z) = x + \cos^2 z$, $\boldsymbol{\sigma}$ as in Exercise 29.
31. $f(x, y, z) = x \cos z$, $\boldsymbol{\sigma}(t) = t\mathbf{i} + t^2\mathbf{j}$; $0 \leqslant t \leqslant 1$.
32. $f(x, y, z) = \exp\sqrt{z}$, $\boldsymbol{\sigma}(t) = (1, 2, t^2)$; $0 \leqslant t \leqslant 1$.
33. $f(x, y, z) = yz$, $\boldsymbol{\sigma}(t) = (t, 3t, 2t)$; $1 \leqslant t \leqslant 3$.
34. $f(x, y, z) = (x + y)/(y + z)$, $\boldsymbol{\sigma}(t) = (t, \frac{2}{3}t^{3/2}, t)$; $1 \leqslant t \leqslant 2$.

★35. Show that the value of the line integral of a scalar function over a parametric curve, defined after Exercise 28, is unchanged if the curve is reparametrized.

18.2 Path Independence

The line integral of a gradient vector field depends only on the endpoints of the curve.

We saw in the last section that the line integral of a vector field along a curve from a point A to a point B depends not just on A and B, but on the path of integration itself. There is, however, an important class of vector fields for which line integrals are path independent. In this section and the next, we shall give several different ways to recognize and use such vector fields.

A vector field $\boldsymbol{\Phi}(x, y, z)$ defined on some domain D in space (or a vector field $\boldsymbol{\Phi}(x, y)$ defined on a domain in the plane) is called *conservative* if, whenever C_1 and C_2 are curves in D with the same endpoints, the line integrals $\int_{C_1} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ and $\int_{C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ are equal.

Our first observation is that a vector field on D is conservative if and only if its integral around every *closed* curve in D is zero. (A conservative force field is thus one in which no net work is done, i.e., energy is conserved if a particle goes around a closed path.)

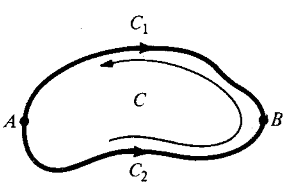

Figure 18.2.1. C_1 and C_2 have the same endpoints when $C = C_2 + (-C_1)$ is closed.

To justify this observation, we consider Figure 18.2.1, which can be interpreted in two different ways. First of all, if C_1 and C_2 are given curves from A to B, then $C = C_2 + (-C_1)$ is a closed curve (from A to A). If $\boldsymbol{\Phi}$ has the property that its integral around every closed curve is zero, then by formulas (7) and (8) in Section 18.1,

$$\int_{C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} - \int_{C_1} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2+(-C_1)} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = 0;$$

so $\int_{C_1} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$. Since this is true for all curves C_1 and C_2 with common endpoints, $\boldsymbol{\Phi}$ is conservative.

The second way to look at Figure 18.2.1 is to consider the closed curve C as given; the pieces C_1 and C_2 are then manufactured by choosing points A and B on C so that $C = C_1 + (-C_2)$. Now if $\boldsymbol{\Phi}$ is conservative, then

$$0 = \int_{C_2} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} - \int_{C_1} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2+(-C_1)} \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r};$$

so the integral of $\boldsymbol{\Phi}$ around a closed curve C is zero.

The argument just given to connect path independence with integrals around closed curves has many applications in mathematics. A related geometric example is given in Exercise 37.

Conservative Fields

A vector field $\mathbf{\Phi}$ is conservative when the line integral of $\mathbf{\Phi}$ around any closed curve is zero, which is the same as the line integrals of $\mathbf{\Phi}$ being path independent.

Example 1 Let $\mathbf{\Phi}$ be a conservative vector field in the plane. If the line integral of $\mathbf{\Phi}$ along the curve AOB in Figure 18.2.2 equals 3.5, find the integral of $\mathbf{\Phi}$ along the broken lines: (a) ACB, (b) BDA, (c) $ACBDA$.

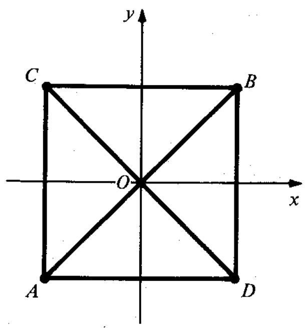

Figure 18.2.2. Paths for Example 1.

Solution (a) Since AOB and ACB have the same endpoints, and $\mathbf{\Phi}$ is conservative, the integral of $\mathbf{\Phi}$ along ACB is 3.5.
(b) BDA has the same endpoints as BOA, which is $-AOB$, so the integral is -3.5.
(c) $ACBDA$ is a closed curve, so the line integral around it is zero. ▲

Example 2 Using examples from the previous section, show that neither of the vector fields $y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ and $-y\mathbf{i} + x\mathbf{j}$ is conservative.

Solution In Example 1, Section 18.1, the line integrals of $y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ along the paths (a) and (c) are different, although the curves have the same endpoints. In Example 5(b), the line integral of $-y\mathbf{i} + x\mathbf{j}$ around a closed curve is not zero, so it cannot be conservative. ▲

Conservative vector fields are easy to integrate because we can replace complicated paths by simple ones, but we still do not know how to recognize these fields. A first step in this direction is the following result.

Theorem: Gradient Vector Fields Are Conservative

If $\mathbf{\Phi} = \nabla f$, then $\mathbf{\Phi}$ is conservative. In fact, if $\mathbf{\Phi} = \nabla f$, then for any curve C from A to B,

$$\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A).$$

To prove this theorem, we let $\boldsymbol{\sigma}(t)$ be a parametrized curve on $[t_1, t_2]$ representing C and going from A to B, so that

$$\int_{C_1} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{\Phi}(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)dt = \int_{t_1}^{t_2} \nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)\, dt$$

By the chain rule for gradients and curves (Section 16.1), $\nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t) = (d/dt)[f(\boldsymbol{\sigma}(t))]$, so

$$\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \frac{d}{dt}\left[f(\boldsymbol{\sigma}(t))\right] dt = f(\boldsymbol{\sigma}(t_2)) - f(\boldsymbol{\sigma}(t_1)) = f(B) - f(A).$$

In the next-to-last equality, we used the fundamental theorem of calculus (Section 4.4). ■

Example 3 Evaluate $\int_C y\,dx + x\,dy$ if C is parametrized by $(t^9, \sin^9(\pi t/2))$, $0 \leqslant t \leqslant 1$.

Solution We recognize the vector field $\boldsymbol{\Phi} = y\mathbf{i} + x\mathbf{j}$ (by guesswork) as the gradient of $f(x, y) = xy$, since

$$f_x = y \quad \text{and} \quad f_y = x.$$

The path of integration goes from $(0,0)$ at $t = 0$ to $(1,1)$ at $t = 1$, so the theorem above gives

$$\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = f(1,1) - f(0,0) = 1. \blacktriangle$$

Note the resemblance of the formula

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A),$$

in the box above to the fundamental theorem of calculus. By analogy with the one-dimensional case, we call a function f such that $\nabla f = \boldsymbol{\Phi}$ an *antiderivative* for f (the term *primitive* is sometimes used). The theorem also shows a big difference between the one-dimensional and multi-dimensional cases: whereas every continuous function of one variable has an antiderivative (see Section 4.5), in several variables, only conservative vector fields *can* have antiderivatives. In fact, every conservative vector field *does* have an antiderivative.

Theorem: Every Conservative Vector Field is a Gradient

If $\boldsymbol{\Phi}(x, y)$ is a vector field defined in a region D, and $\boldsymbol{\Phi}$ is conservative, then there is a function f defined on D such that $\boldsymbol{\Phi} = \nabla f$.

To prove this theorem, we must construct an antiderivative for any given conservative vector field $\boldsymbol{\Phi}$. We do so by "integrating $\boldsymbol{\Phi}$" as follows. Arbitrarily choose a point O in the domain D of the vector field. For every point A in D, we define $f(A)$ by integrating $\boldsymbol{\Phi}$ along some path from O to A. (The word "path" is synonymous with "curve.") Since $\boldsymbol{\Phi}$ is conservative, we get the same result no matter what path we choose. (We are implicitly assuming that D is *connected*, i.e., that it consists of just one piece, so that every point may be joined to O by some path in D. If D has several pieces, the argument is still valid if we work with one piece at a time.)

To show that $\nabla f = \boldsymbol{\Phi}$, we shall show that $\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ for every curve C. Once we have done this, it follows that

$$\int_C \left[\boldsymbol{\Phi}(\mathbf{r}) - \nabla f(\mathbf{r})\right] \cdot d\mathbf{r} = 0$$

for all C; but this implies that $\boldsymbol{\Phi}(\mathbf{r}) - \nabla f(\mathbf{r})$ is identically zero (if not, its integral over a suitably chosen short straight path would be nonzero—see Exercise 36), so $\boldsymbol{\Phi} = \nabla f$.

To prove that $\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ for any path C, we refer to Figure 18.2.3. Since we know that $\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A)$ by the previous theorem, we need to show that $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A)$. Now $f(B) = \int_{C_2} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ and $f(A) = \int_{C_1} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$. Thus $f(B) - f(A) = \int_{(-C_1)+C_2} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ since $(-C_1) + C_2$ and C have the same endpoints and $\mathbf{\Phi}$ is conservative. ■

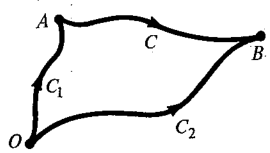

Figure 18.2.3. C and $(-C_1) + C_2$ are both paths from A to B.

Example 4 Calculate the work done in moving a mass m from a distance r_1 out to a distance r_2 in the gravitational field of mass M which produces the force field $\mathbf{F} = -(GMm/r^3)\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the mass M is located at the origin.

Solution From Example 4, Section 16.1, $\mathbf{F} = -\nabla V$, where $V = -GMm/r$. Let the curve C join the points A and B at distances r_1 and r_2 from the origin. The work done by $\mathbf{F}$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla V \cdot d\mathbf{r} = -(V(B) - V(A))$$

$$= V(A) - V(B) = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

The work done in doing this move is therefore $GMm(1/r_1 - 1/r_2)$. If the signs confuse you, note that a spacecraft moving a payload from a distance r_1 to a distance $r_2 > r_1$ does *positive* work. ▲

We still do not know how to tell whether a vector field is conservative just by looking at a formula like $\mathbf{\Phi}(x, y, z) = \cos yz\mathbf{i} + e^x\mathbf{j} + \mathbf{k}$, nor do we have a computationally efficient way of finding antiderivatives. The following theorem provides the method we need. For simplicity, we present it for two variables; the analogous result for three variables is given in Exercise 30.

Theorem: The Cross-Derivative Test

A vector field $\mathbf{\Phi}(x, y) = a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$ defined on the whole plane is conservative if and only if $a_y = b_x$.

The following proof gives an explicit method for finding an antiderivative of $\mathbf{\Phi}$. First, we note from pages 896 and 897 that $\mathbf{\Phi}$ is conservative if and only if it is of the form ∇f, i.e., if and only if we can solve the simultaneous equations[1]

$$f_x = a, \tag{1}$$

$$f_y = b. \tag{2}$$

We notice that if (1) and (2) have a solution f, then (1) implies $f_{xy} = a_y$ while (a) implies $f_{yx} = b_x$. By the equality of mixed partial derivatives (Section 15.1), $f_{xy} = f_{yx}$, so we must have $a_y = b_x$ whenever the vector field $a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$ is conservative.

[1] These are called *partial differential equations*, since they involve partial derivatives of the unknown function f.

To prove that $a_y = b_x$ implies that $a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$ is a gradient, we begin by solving equation (1). We do this by integrating (1) with respect to x, thinking of y as being held fixed just as we do in partial differentiation, to get a trial solution $\bar{f}(x, y)$ of our equations. (You may wish to skip ahead and read Example 5 before finishing this proof.) For instance, we may take

$$\bar{f}(x, y) = \int_0^x a(t, y)\,dt.$$

The choice of 0 as the starting x value for integration was arbitrary, and indeed we get other solutions of equation (1) by adding any function $g(y)$, since the x derivative of $f(x, y) = \bar{f}(x, y) + g(y)$ is still $a(x, y)$. The function $g(y)$ plays the role of the "arbitrary constant" in this partial indefinite integration; what we must do next is to choose it so that equation (2) as well as (1) will be satisfied. With $f(x, y) = \bar{f}(x, y) + g(y)$, equation (2) becomes

$$\bar{f}_y(x, y) + g'(y) = b(x, y),$$

i.e., $g'(y) = b(x, y) - \bar{f}_y(x, y)$. (3)

We have written $g'(y)$ rather than $g_y(y)$ since g is a function of *one* variable.

Equation (3) can be solved by ordinary integration with respect to y provided that the right-hand side is a function of y alone, i.e., if the x derivative of $b - \bar{f}_y$ is zero; but $(b - \bar{f}_y)_x = b_x - \bar{f}_{yx} = b_x - \bar{f}_{xy} = b_x - a_y$, which is zero by the hypothesis $a_y = b_x$. ■

Example 5 Show that the vector field $(2x + 3y^3)\mathbf{i} + (9xy^2 + 2y)\mathbf{j}$ is conservative, and find an antiderivative.

Solution With $a(x, y) = 2x + 3y^3$ and $b(x, y) = 9xy^2 + 2y$, we have $a_y(x, y) = 9y^2$ and $b_x(x, y) = 9y^2$, so the cross-derivative test shows that the given field is conservative. To find an antiderivative f, we first solve the equation

$$f_x(x, y) = 2x + 3y^3$$

by integrating with respect to x to obtain the trial solution $x^2 + 3xy^3$. To this we may add an arbitrary function $g(y)$ without destroying the property that its x derivative is $2x + 3y^3$. Thus, our trial solution is $f(x, y) = x^2 + 3xy^3 + g(y)$. Now the equation $f_y = b$ becomes

$$9xy^2 + g'(y) = 9xy^2 + 2y,$$

so $g'(y) = 2y$. (The fact that the equation for $g'(y)$ does not involve x is a consequence of the condition $a_y = b_x$.) A solution of $g'(y) = 2y$ is $g(y) = y^2$, so we may take $f(x, y) = x^2 + 3xy^3 + y^2$ as our antiderivative. ▲

Example 6 Show that the vector field $x^2\mathbf{i} + xy\mathbf{j}$ is not conservative.

Solution With $a(x, y) = x^2$ and $b(x, y) = xy$, we have $a_y = 0$ and $b_x = y$. Since these cross-derivatives are unequal, the field cannot be conservative. ▲

The next example illustrates the importance of the condition in the previous theorem that the vector field be defined in the *entire* plane.

Example 7 (a) Show that the vector field

$$\mathbf{\Phi}(x, y) = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$$

is not conservative by integrating it around the circle $x^2 + y^2 = 1$.

(b) Verify that the cross-derivative condition $a_y = b_x$ is nevertheless satisfied for this vector field.
(c) What is going on here?

Solution (a) Parametrizing the unit circle C by $\boldsymbol{\sigma}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ gives

$$\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} \left(\frac{-\sin t}{\sin^2 t + \cos^2 t}\mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\mathbf{j} \right) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j})\, dt$$

$$= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t}\, dt = 2\pi.$$

Since this is not zero, $\boldsymbol{\Phi}$ is not conservative.
(b) With $a(x, y) = -y/(x^2 + y^2)$ and $b(x, y) = x/(x^2 + y^2)$, we have

$$a_y(x, y) = \frac{(x^2 + y^2)(-1) - (-y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$b_x(x, y) = \frac{(x^2 + y^2)(1) - (x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

so the cross-derivative condition is satisfied.
(c) The vector field $\boldsymbol{\Phi}(x, y)$ is not defined at the origin, so the cross-derivative test does not apply. ▲

Exercises for Section 18.2

Let $\boldsymbol{\Phi}$ be a conservative vector field in the plane. Suppose that the integral of $\boldsymbol{\Phi}$ along AOF is 3, along OF is 2, and along AB is -5. Compute the integral of $\boldsymbol{\Phi}$ along the paths in Exercises 1–4.

1. $AODEF$
2. $FEDO$
3. $BOEF$
4. $BAODEF$

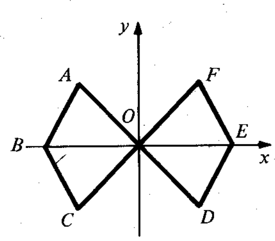

Figure 18.2.4. Paths for Exercise 4.

In Exercises 5–8, evaluate the integral of the given vector field around the given closed curve to demonstrate that it is not conservative.

5. $\boldsymbol{\Phi} = y\mathbf{i} + y\mathbf{j} + \mathbf{k}$, C the path consisting of straight lines joining $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,0,0)$.
6. $\boldsymbol{\Phi} = 2\mathbf{i} + x\mathbf{j}$, C the path in the plane consisting of straight lines from $(0,0)$ to $(1,1)$ to $(-1,1)$ to $(0,0)$.
7. $\boldsymbol{\Phi} = 3\mathbf{i} + x\mathbf{j}$, C the unit circle $x^2 + y^2 = 1$.
8. $\boldsymbol{\Phi} = y\mathbf{i} - xy\mathbf{j}$, C the unit circle $x^2 + y^2 = 1$.

In Exercises 9–12, recognize the vector field as a gradient and use this to evaluate the given integral.

9. $\int_C 2xy\, dx + x^2\, dy$, C parametrized by $x = \cos 8t$, $y = 5\sin 16t$, $0 \leqslant t \leqslant \pi/4$.
10. $\int_C ye^{xy}\, dx + xe^{xy}\, dy$, C parametrized by $x = 5t^3$, $y = -t^3$, $-1 \leqslant t \leqslant 1$.
11. $\int_C 3x^2y^2\, dx + 2x^3y\, dy$, C parametrized by $x = 3t^2 + 1$, $y = 2t$, $0 \leqslant t \leqslant 1$.
12. $\int_C y\sin(xy)\, dx + x\sin(xy)\, dy$, C parametrized by $x = \cos 2t$, $y = 3\sin 2t$, $0 \leqslant t \leqslant \pi/2$.

13. A certain force field exerted on a mass m is given by $\mathbf{F} = -(JMm/r^5)\mathbf{r}$. Find the work done in moving the mass m from a distance r_1 out to a distance $r_2 > r_1$.
14. The mass of the earth is approximately 6×10^{27} grams; the mass of the sun is 330,000 times as much. The gravitational constant (in units of grams, seconds, and centimeters) is 6.7×10^{-8}. The distance of the earth from the sun is about 1.5×10^{12} centimeters. Compute, approximately, the work necessary to increase the distance of the earth from the sun by 1 centimeter.
15. Suppose that the kinetic energy of a particle which moves in a circular path increases after it makes one circuit. Can the force field governing the particle's motion be conservative?
16. In the earth's gravitational field, show that if a mass is taken on a journey, however complicated, the net amount of work done against the

gravitational field is zero, provided the mass ends up in the same spot it began, with the same speed.

17. Is $2xy\mathbf{i} + (x^2 + \cos y)\mathbf{j}$ conservative? If so, find an antiderivative.
18. Is $x^2y\mathbf{i} + (\frac{1}{3}x^3 + ye^y)\mathbf{j}$ conservative? If so, find an antiderivative.
19. Is $4x\cos^2(y/2)\mathbf{i} - x^2\sin y\mathbf{j}$ conservative? If so, find an antiderivative.
20. Is $2xy\sin(x^2y)\mathbf{i} + (e^y + x^2\sin(x^2y))\mathbf{j}$ conservative? If so, find an antiderivative.
21. Consider the vector field in Example 7. Show that if we restrict its domain to be those (x, y) with $y > 0$, then it is conservative. (Show that $f(x, y) = \tan^{-1}(y/x)$ is an antiderivative.)
22. In Exercise 21, what prevents f from being an antiderivative for $\mathbf{\Phi}$ on the whole plane minus the origin?

In Exercises 23–26, recognize $\mathbf{\Phi}$ as a gradient and compute the work done in moving a particle along the given path with the given force.

23. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, t)$; $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
24. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, 0)$; $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
25. Same as Exercise 24 but with $0 \leqslant t \leqslant \pi$.
26. $\boldsymbol{\sigma}(t) = (a\cos t, 0, b\sin t)$; $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(r) = -\mathbf{r}/\|\mathbf{r}\|^3$.

27. Let $\mathbf{\Phi}(x, y, z) = (z^3 + 2xy)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Show that the integral of $\mathbf{\Phi}$ around the circumference of the unit square $[0, 1] \times [0, 1]$ in the xy plane is zero by:
    (a) Evaluating directly.
    (b) Demonstrating that $\mathbf{\Phi}$ is the gradient of some function f.
28. Let
$$f(x, y, z) = e^x\cos(yz + x^8) + [\sin(yz)]\ln(1 + x^2).$$
Show that
$$\int_{C_1} \nabla f(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \nabla f(\mathbf{r}) \cdot d\mathbf{r},$$
where C_1 is the straight line joining $(0, 1, 0)$ to $(1, 1, 0)$ and C_2 is the curve parametrized by $(\sin t, \cos 4t, \sin 4t)$; $0 \leqslant t \leqslant \pi/2$.
29. If $f(x, y, z) = x^3 - y^3 + \sin(\pi yz/2)$, evaluate $\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r}$ along the curve
$$\boldsymbol{\sigma}(t) = \left(\frac{t^3}{1 + t^2}, \sin\frac{\pi}{4}t^5, t^2 + 2\right); \qquad 0 \leqslant t \leqslant 1.$$
★30. Extend the cross-derivative test to vector fields in space: show that $\mathbf{\Phi}(x, y, z) = a(x, y, z)\mathbf{i} + b(x, y, z)\mathbf{j} + c(x, y, z)\mathbf{k}$ is conservative if and only if $c_y = b_z$, $c_x = a_z$ and $a_y = b_x$.

Use the cross-derivative test in Exercise 30 to determine whether each of the vector fields in Exercises 31–34 is conservative. If it is, find an antiderivative.

★31. $2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + y\mathbf{k}$
★32. $xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
★33. $e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$
★34. $\cos(xy)\mathbf{i} + yx\mathbf{j} - \sin(yz)\mathbf{k}$

★35. Show that any two antiderivatives of a vector field in the plane or space differ by a constant.
★36. Show that only the zero vector field is *totally path independent* in the sense that its integrals over all paths, even with different endpoints, are equal.
★37. *Two-color problem.* Several intersecting circles are drawn in the plane. Show that the resulting "map" can be colored with two colors in such a way that adjacent regions have different colors (as in Fig. 18.2.5). [*Hint*: First show that every closed curve crosses the union of the circles an even number of times. Then divide the regions into two classes according to whether an arc from the region to a fixed point crosses the circles an even or odd number of times. Compare your argument with the proof of the basic properties of conservative vector fields.]

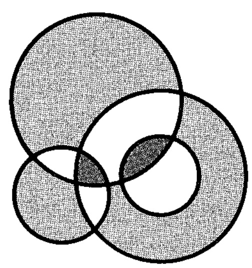

Figure 18.2.5. Adjacent regions have opposite colors.

18.3 Exact Differentials

Gradient vector fields correspond to exact differentials.

In the preceding section, we established the cross-derivative test: a vector field $\mathbf{\Phi}(x, y) = a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$ defined on the whole plane is the gradient of some function if and only if $a_y = b_x$. In this section, we shall use this result to solve a class of differential equations called exact equations. In doing this, it is convenient to use the notation of differential forms, so we begin by summarizing this notation.

Just as the differential notation $\int f(x)\,dx$ was convenient for functions of one variable, we have seen that the notation

$$\int_C \mathbf{\Phi}(x,y)\cdot d\mathbf{r} = \int_C a(x,y)\,dx + b(x,y)\,dy$$

is a good one for the line integral of a vector field $\mathbf{\Phi}(x,y) = a(x,y)\mathbf{i} + b(x,y)\mathbf{j}$. The expression $a(x,y)\,dx + b(x,y)\,dy$ is called a *differential form*; such an expression is often written as $P\,dx + Q\,dy$, where $P = a(x,y)$ and $Q = b(x,y)$. The differential form $P\,dx + Q\,dy$ is called *exact* if there is a function $u = f(x,y)$ such that $P = \partial u/\partial x$ and $Q = \partial u/\partial y$, so $P\,dx + Q\,dy = (\partial u/\partial x)\,dx + (\partial u/\partial y)\,dy$. We call $u = f(x,y)$ an *antiderivative* of the differential form. Using this notation, we can rewrite the cross-derivative test as follows.

Cross-Derivative Test for Differential Forms

Let $P\,dx + Q\,dy$ be a differential form defined in the plane. If P and Q have continuous partials and $\partial P/\partial y = \partial Q/\partial x$, then $P\,dx + Q\,dy$ is exact—that is, there is a function $u = f(x,y)$ such that $P = \partial u/\partial x$ and $Q = \partial u/\partial y$. (In other words, the vector field $P\mathbf{i} + Q\mathbf{j}$ is the gradient of f.)

Given a differential form $P\,dx + Q\,dy$ satisfying $\partial P/\partial y = \partial Q/\partial x$, there are two ways of finding $u = f(x,y)$. The first is the method we used in Example 5, Section 18.2.

Method 1. If $P = \partial u/\partial x$, integrate to give $u = \int P\,dx + g(y)$, where $g(y)$ is a function of y to be determined. Differentiate u with respect to y and equate your answer to Q. This will give an equation for $g'(y)$ which may be integrated to yield $g(y)$ and hence u.

Method 2. If C is a curve from (x_0, y_0) to (x,y) and if $\mathbf{\Phi}(x,y) = \nabla f(x,y)$ then from p. 896,

$$f(x,y) - f(x_0, y_0) = \int_C \mathbf{\Phi}\cdot d\mathbf{r}.$$

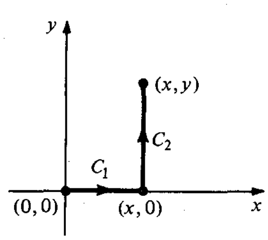

Figure 18.3.1. The path used to construct $u = f(x,y)$.

Choose C to be the curve from $(0,0)$ to (x,y) shown in Figure 18.3.1. We can also adjust f by a constant so that $f(0,0) = 0$. This leads to the explicit formula

$$f(x,y) = \int_0^x a(t,0)\,dt + \int_0^y b(x,t)\,dt \tag{1}$$

for $u = f(x,y)$, where $P = a(x,y)$ and $Q = b(x,y)$. In (1), $\int_0^x a(t,0)\,dt$ is the integral of $P\,dx + Q\,dy$ along C_1, while $\int_0^y b(x,t)\,dt$ is the integral along C_2.

Example 1 Is $(2xy\cos y + x^2)\,dx + (x^2\cos y - x^2y\sin y)\,dy$ an exact differential? If so, find an antiderivative; that is, find u such that $\partial u/\partial x = 2xy\cos y + x^2$ and $\partial u/\partial y = x^2\cos y - x^2y\sin y$.

Solution To test for exactness, we let $P = 2xy\cos y + x^2$ and $Q = x^2\cos y - x^2y\sin y$ and compute that

$$\frac{\partial P}{\partial y} = -2xy\sin y + 2x\cos y = \frac{\partial Q}{\partial x},$$

and so a u exists. To find u, we can use either of the two methods above.

Method 1. We must have

$$\frac{\partial u}{\partial x} = P \quad\text{and}\quad \frac{\partial u}{\partial y} = Q,$$

Hence

$$\frac{\partial u}{\partial x} = 2xy\cos y + x^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = x^2\cos y - x^2y\sin y.$$

Integrating the first equation with respect to x gives $u = x^2y\cos y + x^3/3 + g(y)$; the "constant" of integration g must be a function of y alone. Differentiating with respect to y gives $\partial u/\partial y = x^2\cos y - x^2y\sin y + g'(y)$. Since $\partial u/\partial y = x^2\cos y - x^2y\sin y$, $g'(y) = 0$, and so we may take $g(y) = 0$. Thus $u = x^3/3 + x^2y\cos y$.

Method 2. We use formula (1) above:

$$f(x, y) = \int_0^x a(t,0)\,dt + \int_0^y b(x,t)\,dt,$$

where $P = a(x, y)$ and $Q = b(x, y)$. In this case, $a(x, y) = 2xy\cos y + x^2$ and $b(x, y) = x^2\cos y - x^2y\sin y$, so we get

$$f(x, y) = \int_0^x t^2\,dt + \int_0^y (x^2\cos t - x^2t\sin t)\,dt.$$

Evaluating the last integral by integrating by parts gives

$$\begin{aligned} u = f(x, y) &= \frac{x^3}{3} + \int_0^y x^2\cos t\,dt - \int_0^y x^2\cos t\,dt + x^2t\cos t\Big|_0^y \\ &= \frac{x^3}{3} + x^2y\cos y, \end{aligned}$$

which agrees with the answer using Method 1. We check that $\partial u/\partial x = P$ and $\partial u/\partial y = Q$, as required. ▲

Exact Differentials

To determine whether there is a function $u = f(x, y)$ such that $\partial u/\partial x = P = a(x, y)$ and $\partial u/\partial y = Q = b(x, y)$, check whether $\partial P/\partial y = \partial Q/\partial x$.

If so, f may be constructed by formula (1) or by integrating: $u = \int P\,dx + g(y)$; equating $\partial u/\partial y$ and Q determines $g'(y)$ and hence, by integration, $g(y)$.

If $\partial P/\partial y \neq \partial Q/\partial x$, no such f can exist.

Example 2 Show that, in place of formula (1), we can also choose

$$f(x, y) = \int_0^1 \left[xa(tx, ty) + yb(tx, ty) \right] dt.$$

Solution The function f is the integral of $P\,dx + Q\,dy$ along any path from $(0,0)$ to (x, y). (In the proof, we used paths parallel to the axes.) The expression $\int_0^1 [xa(tx,ty) + yb(tx,ty)]\,dt$ is just the integral of $P\,dx + Q\,dy$ along the path (tx, ty), $0 \leqslant t \leqslant 1$; that is, the straight line segment joining $(0,0)$ to (x, y). ▲

In Chapters 8 and 12, we studied a number of useful classes of differential equations. A new class may be solved by the methods of this section. A differential equation of the form

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0 \tag{2}$$

will be called *exact* if the corresponding differential form $P\,dx + Q\,dy$ is exact.

These equations may be solved as follows: there is a function $f(x, y)$ such that $P = f_x$ and $Q = f_y$. The curves

$$f(x, y) = C \tag{3}$$

for C a constant are solutions of (2) assuming that they define y as a function of x. Indeed, differentiating (3) implicitly by the chain rule, we get

$$f_x + f_y \frac{dy}{dx} = 0, \quad \text{i.e.,} \quad P + Q\frac{dy}{dx} = 0.$$

Exact Differential Equations

To test if the equation

$$P + Q\frac{dy}{dx} = 0$$

is exact, see if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

To solve an exact equation $P + Q(dy/dx) = 0$, find $f(x, y)$ such that

$$P = f_x \quad \text{and} \quad Q = f_y.$$

(See the preceding box.) The solutions are $f(x, y) = C$ for any constant C.

Example 3 Find the solution of the differential equation

$$2xy\cos y + x^2 + (x^2\cos y - x^2y\sin y)\frac{dy}{dx} = 0$$

that passes through $(1, 0)$.

Solution By Example 1, this equation is exact with

$$f(x, y) = \frac{x^3}{3} + x^2y\cos y.$$

The solution is thus $(x^3/3) + x^2y\cos y = C$. Since $y = 0$ when $x = 1$, $C = \frac{1}{3}$. Thus our solution is given implicitly by

$$\frac{x^3}{3} + x^2y\cos y = \frac{1}{3}. \blacktriangle$$

Example 4 (a) Let $(x, y) = (e^{t-1}, \sin(\pi/t))$, $1 \leqslant t \leqslant 2$, be a parametrization of the curve C. Calculate $\int_C 2x\cos y\,dx - x^2\sin y\,dy$.

(b) Find the solution of $2x\cos y = x^2\sin y(dy/dx)$ that satisfies $y(3) = 0$.

Solution (a) The curve C goes from $(1, 0)$ to $(e, 1)$. Since $\partial(2x\cos y)/\partial y = -2x\sin y = \partial(-x^2\sin y)/\partial x$, the integrand is exact. Thus we can replace C by any curve having the same endpoints, in particular by the polygonal path from $(1, 0)$ to $(e, 0)$ to $(e, 1)$. Thus the line integral must be equal to

$$\int_1^e 2t\cos 0\,dt + \int_0^1 -e^2\sin t\,dt = (e^2 - 1) + e^2(\cos 1 - 1)$$
$$= e^2\cos 1 - 1.$$

Alternatively, using the antiderivative $u = f(x, y) = x^2\cos y$, which may be

found by the methods of Example 1,

$$\int_C 2x\cos y\,dx - x^2\sin y\,dy = \int_C \nabla f(\mathbf{r})\cdot d\mathbf{r} = f(e,1) - f(1,0)$$

$$= e^2\cos 1 - 1.$$

(b) The general solution is $x^2\cos y = C$. Since $y = 0$ when $x = 3$ we have $x^2\cos y = 9$; that is, $y = \cos^{-1}(9/x^2)$. ▲

Example 5 (a) Verify that $\partial P/\partial y = \partial Q/\partial x$ and find $u = f(x, y)$ such that $P = \partial u/\partial x$ and $Q = \partial u/\partial y$ in each case:
(i) $P = x^2 + y^3$; $Q = 3xy^2 + 1$.
(ii) $P = y\cos x$; $Q = y + \sin x$.

(b) Solve the following differential equations with the given conditions:
(i) $x^2 + y^3 + (3xy^2 + 1)\,dy/dx = 0$; $y(0) = 1$.
(ii) $y\cos x + (y + \sin x)\,dy/dx = 0$; $y(\pi/4) = 1$.

Solution (a) (i) We have $P_y = 3y^2$ and $Q_x = 3y^2$, and so $P_y = Q_x$. Let $f(x, y)$ be such that $f_x = x^2 + y^3$ and $f_y = 3xy^2 + 1$. Then $f(x, y) = \int f_x\,dx = x^3/3 + xy^3 + g(y)$. Then $f_y(x, y) = 3xy^2 + g'(y) = 3xy^2 + 1$ implies $g'(y) = 1$, so $g(y) = y + C$. Thus, $f(x, y) = x^3/3 + xy^3 + y + C$.

(ii) We have $\partial P/\partial y = \cos x$ and $\partial Q/\partial x = \cos x$; thus $\partial P/\partial y = \partial Q/\partial x$. Integrating, $f(x, y) = \int f_x\,dx = y\sin x + g(y)$; $f_y = \sin x + g'(y) = \sin x + y$ implies $g'(y) = y$ or $g(y) = y^2/2 + C$. We thus have $f(x, y) = y\sin x + y^2/2 + C$.

(b) (i) The differential equation $x^2 + y^3 + (3xy^2 + 1)(dy/dx) = 0$ is exact by our calculation in part (a)(i). From that calculation we find that $f(x, y) = x^3/3 + y^3x + y = K$ is a solution. Applying the condition $y(0) = 1$ gives $K = 1$. Thus the solution is $x^3/3 + y^3x + y = 1$.

(ii) Similarly, using the result of part (a)(ii), we get the solution $f(x, y) = y\sin x + y^2/2 = K$. Applying the condition that $y(\pi/4) = 1$ gives $K = (\sqrt{2} + 1)/2$. Thus the solution is $y\sin x + y^2/2 = (\sqrt{2} + 1)/2$. ▲

A differential equation

$$M + N\frac{dy}{dx} = 0 \tag{4}$$

which is not exact may sometimes be made exact if we multiply it through by a function $\mu(x, y)$. The equivalent equation is $\mu M + \mu N\,dy/dx = 0$, which is exact if $\partial(\mu M)/\partial y = \partial(\mu N)/\partial x$. This is a condition on μ, which can be written as

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x\,. \tag{5}$$

Any function $\mu(x, y)$ satisfying equation (5) is called an *integrating factor* for the original equation (4).

In general, it is not easy to solve (5), but occasionally it is possible to solve it with a μ which is a function of x alone, i.e., $\mu(x, y) = \mu(x)$, in which case $\mu_x = \mu'$ and $\mu_y = 0$. In this case (5) becomes $\mu M_y = \mu' N + \mu N_x$ or

$$\frac{M_y - N_x}{N} = \frac{\mu'}{\mu} = (\ln\mu)'.$$

This can be solved for μ if $(M_y - N_x)/N$ is a function of x alone, in which case $\ln \mu = \int[(M_y - N_x)/N]\,dx$, and so the integrating factor is

$$\mu = \exp\left[\int \frac{M_y - N_x}{N}\,dx\right].$$

Having found an integrating factor, one may now solve the exact equation $\mu M + \mu N\,dy/dx = 0$ by the methods described earlier in this section. The following example shows how linear equations may be solved by this method (compare this approach with the methods of Section 8.6).

Example 6 (a) Solve the equation $x\,dy/dx - x^5 + x^3y - y = 0$.
(b) Solve the linear equation $dy/dx + P(x)y + Q(x) = 0$ using the method of integrating factors.

Solution (a) Comparing the equation with (4), we see that $M = -x^5 + x^3y - y$ and $N = x$. Thus $M_y = x^3 - 1$ and $N_x = 1$. Hence $(M_y - N_x)/N = (x^3 - 1 - 1)/x = x^2 - 2/x$. Therefore, the integrating factor is $\mu = \exp[\int(x^2 - 2/x)\,dx] = \exp(x^3/3 - 2\ln x) = \exp(x^3/3)/x^2$. According to formula (1) applied to $\mu M\,dx + \mu N\,dy$,

$$\begin{aligned}
F(x,y) &= \int_0^x a(t,0)\,dt + \int_0^y b(x,t)\,dt\\
&= \int_0^x \frac{\exp(t^3/3)}{t^2}(-t^5)\,dt + \int_0^y \frac{\exp(x^3/3)}{x^2}\,x\,dt\\
&= -\int_0^x t^3\exp\left(\frac{t^3}{3}\right)dt + \frac{y}{x}\exp(x^3/3) = C.
\end{aligned}$$

If we rearrange and use the indefinite integral notation

$$\int_0^x t^3\exp\left(\frac{t^3}{3}\right)dt + C = \int x^3\exp\left(\frac{x^3}{3}\right)dx,$$

we get

$$y = x\exp\left(\frac{-x^3}{3}\right)\int x^3\exp\left(\frac{x^3}{3}\right)dx.$$

(b) Here, $M = P(x)y + Q(x)$, and so $M_y = P(x)$; $N = 1$, so $N_x = 0$. Thus $(M_y - N_x)/N = P(x)$. Therefore, the integrating factor is $\mu = \exp(\int P(x)\,dx)$. Applying (1) to $\mu M\,dx + \mu N\,dy$, we find that the solution is

$$\begin{aligned}
f(x,y) &= \int_0^x a(t,0)\,dt + \int_0^y b(x,t)\,dt\\
&= \int_0^x \exp\left(\int P(t)\,dt\right)Q(t)\,dt + \int_0^y \exp\left(\int P(x)\,dx\right)dt\\
&= \int \exp\left(\int P(x)\,dx\right)Q(x)\,dx + y\exp\left(\int P(x)\,dx\right) = C.
\end{aligned}$$

Absorbing the constant into the first integration and solving for y gives

$$y = -e^{-\int P(x)\,dx}\left[\int Q(x)e^{\int P(x)\,dx}\,dx\right].$$

(The methods of Section 8.6 yield the same answer.) ▲

Exercises for Section 18.3

1. Is $[2x + e^{xy}(xy+1)]\,dx + x^2e^{xy}\,dy$ an exact differential? If so, find an antiderivative.
2. Is $(\cos xy - xy\sin xy)\,dx - (x^2\sin xy)\,dy$ an exact differential? If so, find an antiderivative.
3. Is there a function $u = f(x,y)$ such that $\partial u/\partial x = x\sqrt{x^2y^2+1}$ and $\partial u/\partial y = y\sqrt{x^2y^2+1}$? If so, find it.
4. Is there a function u of (x,y) such that $\partial u/\partial x = 2x\cos y + \cos y$ and $\partial u/\partial y = -x^2\sin y - x\sin y$? If so, find it.

In Exercises 5–8, determine which differentials are exact.

5. $x^2y\,dx + (x^3y/3)\,dy$
6. $x^2\,dx + (x^3/3)\,dy$
7. $x^2y\,dx + (x^3/3)\,dy$
8. $xy^2\,dx + (y^3/3)\,dy$
9. Show that we can use
$$\hat{f}(x,y) = \int_0^y b(0,t)\,dt + \int_0^x a(t,y)\,dt$$
in place of formula (1).
10. Use the path $(\sqrt{t}\,x, tx)$, $0 \leqslant t \leqslant 1$ to find another formula (other than those in Example 2 and Exercise 9) that may be used in place of (1).
11. Let C_1 be parametrized by $(t^2+1, t\sin(\pi t/8))$, $0 \leqslant t \leqslant 4$, and C_2 by $((t^2+1)\cos\pi t, t)$ $0 \leqslant t \leqslant 4$. Determine whether $\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$, where $P\,dx + Q\,dy = \frac{1}{2}(x^4+y^2)\,dx + (xy+e^y)\,dy$.
12. Let $P = \ln(x^2+1) - 2xe^{-y}$ and $Q = x^2e^{-y} - \ln(y^2+1)$. Determine whether $\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$, where C_1 and C_2 are two curves with the same endpoints.

Solve the differential equations satisfying the given conditions in Exercises 13–16.

13. $ye^x + e^y + (xe^y + e^x)\,dy/dx = 0$, $y(0) = 2$.
14. $e^y + (xe^y + 2y)\,dy/dx = 0$, $y(0) = 1$.
15. $3x^2 + 2xy + (x^2+y^2)\,dy/dx = 0$, $y(1) = 2$.
16. $\cos y\sin x + \sin y\cos x\,dy/dx = 0$, $y(\pi/2) = 1$.

Determine which of the equations in Exercises 17–20 are exact and solve the ones that are.

17. $x\,dy/dx + y + x^2y^2 - 1 = 0$.
18. $y - x^3 + (x + y^3)\,dy/dx = 0$.
19. $xy^2 + 3x^2y + (x+y)x^2\,dy/dx = 0$.
20. $e^x\sin y - e^y\sin x\,dy/dx = 0$.
21. (a) Let $(x,y) = (t^3-1, t^6-t)$, $0 \leqslant t \leqslant 1$ parametrize the curve C. Calculate
$$\int_C \left(\frac{2x}{y^2+1}\,dx - \frac{2y(x^2+1)}{(y^2+1)^2}\,dy\right).$$
(b) Find the solution of
$$\frac{x}{y} = \frac{x^2+1}{y^2+1}\frac{dy}{dx}$$
that satisfies $y(1) = 1$.
22. (a) Let $(x,y) = (e^t, e^{t+1})$, $-1 \leqslant t \leqslant 0$ be a parametrization of the curve C. Calculate
$$\int_C \left[\cos(xy^2) - xy^2\sin(xy^2)\right]dx - 2x^2y\sin(xy^2)\,dy.$$
(b) Find the solution of
$$\cos(xy^2) - xy^2\sin(xy^2) = 2x^2y\sin(xy^2)\frac{dy}{dx}$$
that satisfies $y(1) = 0$.
23. Solve the equation in Exercise 21(b) using the method of separation of variables (Section 8.5).
24. Solve the equation $dy/dx = (1-y)/(x+1)$, $y(0) = 2$ using (a) the method of exact equations and (b) separation of variables. Verify that the two answers agree.
25. Find an integrating factor $\mu(x)$ for the equation
$$2y\cos y + x + (x\cos y - xy\sin y)\frac{dy}{dx} = 0.$$
26. Solve the equation $dy/dx = 3x^2y + x$ by finding an integrating factor. (Leave your answer in the form of an integral.)
27. Solve the equation $x\,dy/dx = xy^2 + y$ by using the integrating factor $1/y^2$.
28. Use the integrating factor xy to solve
$$\frac{1}{x} + \frac{1}{y} + \left(\frac{y}{x} + \frac{1}{y}\right)\frac{dy}{dx} = 0.$$
29. Find the equation which must be satisfied by a function μ of y alone if it is to be an integrating factor for $M + N\,dy/dx = 0$.
30. (a) Find a solution of the equation $3x^2y^2 + 2(x^3y + x^3y^3)\,dy/dx = 0$ in the form $u(x,y) =$ constant.
(b) If $y = f(x)$ is a solution of the equation with $f(1) = 1$, what is $f(2)$?

★31. If $P\,dx + Q\,dy$ satisfies $\partial P/\partial y = \partial Q/\partial x$, show that functions $g(y)$ and $h(x)$ can always be found such that $\int P\,dx + g(y) = \int Q\,dy + h(x)$.

★32. Generalize the assertion in Exercise 31 to $P\,dx + Q\,dy + R\,dz$.

18.4 Green's Theorem

A line integral along a closed curve can be converted to a double integral over a region.

Green's[2] theorem relates the line integral of a vector field (or differential form) around a closed curve to the double integral of a certain function over the region bounded by the curve. Among the many applications of this theorem is a formula for the area of a region in terms of a line integral around its boundary.

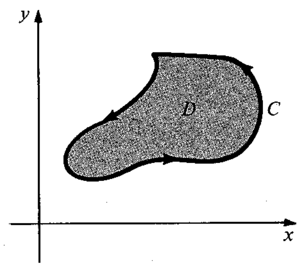

Figure 18.4.1. The boundary C of a region D must be oriented counterclockwise in Green's theorem.

Green's theorem states that

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy, \tag{1}$$

where C is the curve bounding a region D, C is traversed counterclockwise (see Fig. 18.4.1), and P and Q have continuous partial derivatives. We have seen that $P\,dx + Q\,dy$ is exact when $\partial Q/\partial x = \partial P/\partial y$; in this case, Green's theorem is obviously true because each side of (1) is zero.

We shall prove Green's theorem for regions which are of type 1 and 2 and then indicate through examples how the result may be proved for more general regions.

Consider a region of type 1 in the plane, as shown in Fig. 18.4.2. The *boundary* of this region is defined to be the closed curve C which goes once

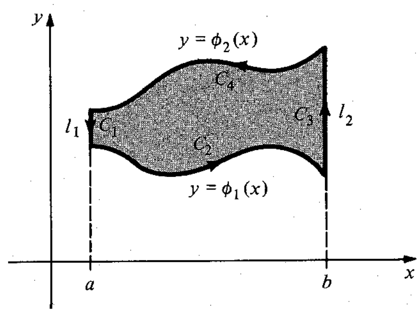

Figure 18.4.2. The boundary of this type 1 region consists of the graphs $y = \phi_1(x)$ and $y = \phi_2(x)$ and segments of the lines l_1 and l_2.

around the region in the counterclockwise direction. If we start at the upper left-hand corner, the boundary curve first traverses the vertical line l_1 (call this C_1), then goes along the graph of ϕ_1 (call this C_2), then up l_2 (call this C_3), and, finally, backwards along the graph of ϕ_2 (call this C_4). The following lemma is a preliminary form of Green's theorem.

Lemma *Let D be a type 1 region as above, with C its bounding curve. Let $P = f(x, y)$ have continuous partial derivatives in D and on C. Then*

$$\int_C P\,dx = -\int\int_D \frac{\partial P}{\partial y}\,dx\,dy. \tag{2}$$

[2] George Green (1793–1841), an English mathematician and physicist, was one of the early investigators of electricity and magnetism. His work on potentials led to what is commonly called Green's theorem (although it may also be due to Cauchy). This theorem, as generalized to three dimensions by Lord Kelvin, Stokes, Gauss, and Ostrogradsky, was crucial to later developments in electromagnetic, gravitational, and other physical theories. Some of these applications are discussed later in this chapter. For more history, see "The history of Stokes' theorem" by V. J. Katz, Mathematics Magazine, **52** (1979) 146–156.

Proof Since the double integral may be evaluated as an iterated integral (see p. 850), we have

$$\int\int_D \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f_y(x, y)\,dy\,dx$$
$$= \int_a^b [f(x,\phi_2(x)) - f(x,\phi_1(x))]\,dx.$$

The latter equality uses the fundamental theorem of calculus.

To compute the line integral, we parametrize each of the segments of C:

C_1: $(x, y) = (a, -t)$; t in $[-\phi_2(a), -\phi_1(a)]$; $dx = 0$, $dy = -dt$.
C_2: $(x, y) = (t, \phi_1(t))$; t in $[a, b]$; $dx = dt$, $dy = \phi_1'(t)\,dt$.
C_3: $(x, y) = (b, t)$; t in $[\phi_1(b), \phi_2(b)]$; $dx = 0$, $dy = dt$.
C_4: $(x, y) = (-t, \phi_2(-t))$; t in $[-b, -a]$; $dx = -dt$, $dy = -\phi_2'(-t)\,dt$.

Now $\int_C P\,dx$ is the sum of the line integrals over the four C_i's. The integrals over C_1 and C_3 are zero, since $dx = 0$ on those curves (x is constant). The integrals over C_2 and C_4 are given by

$$\int_{C_2} P\,dx = \int_a^b f(t,\phi_1(t))\,dt$$

and

$$\int_{C_4} P\,dx = \int_{-b}^{-a} f(-t,\phi_2(-t))(-dt)$$
$$= \int_b^a f(t,\phi_2(t))\,dt \qquad \text{(substituting } -t \text{ for } t\text{)}$$
$$= -\int_a^b f(t,\phi_2(t))\,dt.$$

Thus

$$\int_C P\,dx = \int_a^b f(t,\phi_1(t))\,dt - \int_a^b f(t,\phi_2(t))\,dt$$
$$= -\int_a^b [f(t,\phi_2(t)) - f(t,\phi_1(t))]\,dt$$
$$= -\int\int_D \frac{\partial P}{\partial y}\,dx\,dy. \blacksquare$$

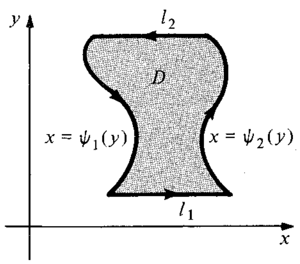

Figure 18.4.3. The boundary of this type 2 region consists of the two curves, $x = \psi_1(y)$ and $x = \psi_2(y)$, and segments of the lines, l_1 and l_2.

In exactly the same way, we can prove that if D is a type 2 region with boundary curve C traversed counterclockwise (Fig. 18.4.3), then for $Q = g(x, y)$ with continuous partial derivatives,

$$\int_C Q\,dy = \int\int_D \frac{\partial Q}{\partial x}\,dx\,dy. \tag{3}$$

If we have a region which is of both types 1 and 2, then equations (2) and (3) are both valid. Adding them yields formula (1).

Green's Theorem *If D is a region of types 1 and 2 with boundary curve C traversed counterclockwise, and if $P = f(x, y)$ and $Q = g(x, y)$ have continuous partial derivatives in D and on C, then*

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Example 1 Verify Green's theorem for $P = x$ and $Q = xy$, where D is the unit disk $x^2 + y^2 \leqslant 1$.

Solution We can evaluate both sides in Green's theorem directly. The boundary C of D is the unit circle parametrized by $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$, so

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} \left[(\cos t)(-\sin t) + \cos t \sin t \cos t \right] dt$$
$$= \left[\frac{\cos^2 t}{2} \right]_0^{2\pi} + \left[-\frac{\cos^3 t}{3} \right]_0^{2\pi} = 0.$$

On the other hand,

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int\int_D y\,dx\,dy,$$

which is zero also, since the contributions from the upper and lower half-circles cancel one another. Thus Green's theorem is verified in this case. ▲

Green's theorem applies as well to many regions other than just those of types 1 and 2. Often one can show this by dividing up the region, as in the following example.

Example 2 Show that Green's theorem is valid for the region D shown in Fig. 18.4.4.

Solution Figure 18.4.5 shows how to divide up D into three regions, D_1, D_2, D_3, each of which is of types 1 and 2. Let C_1, C_2, C_3 be the boundary curves of these regions. Then

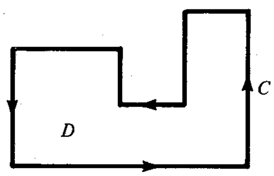

Figure 18.4.4. Is Green's theorem valid for D?

$$\int\int_{D_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{C_i} P\,dx + Q\,dy.$$

The double integral over the D_i's adds up to the double integral over D, so

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy.$$

However, the dotted portions of the boundaries shown in Fig. 18.4.5 are traversed twice in opposite directions; these cancel in the line integrals. Thus we are left with

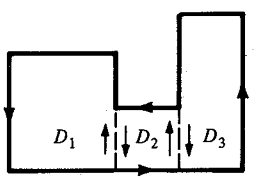

Figure 18.4.5. Breaking a region up into smaller regions, each of which is both type 1 and type 2.

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_C P\,dx + Q\,dy,$$

and so Green's theorem is valid. ▲

Example 2 illustrates a special case of the following procedure for a region D:

(a) Break up D into smaller regions, $D_1, D_2, \ldots, D_n$, each of which is of types 1 and 2.
(b) Apply Green's theorem as proven above to each of $D_1, \ldots, D_n$, and add the resulting integrals.
(c) The line integrals along interior boundaries cancel, leaving the line integral around the boundary of D.

This procedure yields Green's theorem for D. It is plausible that this method applies to any region bounded by piecewise smooth curves, and so we may expect a general form of Green's theorem.

Green's Theorem

If D is a region and C is the boundary of D, oriented as in Fig. 18.4.1, then

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Example 3 Let $\boldsymbol{\Phi}(x, y) = y\mathbf{i} - x\mathbf{j}$ and let C be a circle of radius r traversed counterclockwise. Write $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ as a double integral using Green's theorem. Evaluate.

Solution If $\boldsymbol{\Phi}(x, y) = P\mathbf{i} + Q\mathbf{j}$, then $\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = P\,dx + Q\,dy$. Now we apply Green's theorem to the case where D is the disk of radius r, $Q = -x$, and $P = y$, so $\partial Q/\partial x - \partial P/\partial y = -2$. Thus $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int\int_D (-2)\,dx\,dy = (-2)(\text{area of } D) = -2\pi r^2$. ▲

Example 4 Let C be the boundary of the square $[0, 1] \times [0, 1]$ oriented counterclockwise. Evaluate

$$\int_C (y^4 + x^3)\,dx + 2x^6\,dy.$$

Solution We could, of course, evaluate the integral directly, but it is easier to use Green's theorem. Let D be the unit square (bounded by the lines $x = 0, y = 0$, $x = 1$, and $y = 1$). Then

$$\begin{aligned}\int_C (y^4 + x^3)\,dx + 2x^6\,dy &= \int\int_D \left[\frac{\partial}{\partial x} 2x^6 - \frac{\partial}{\partial y}(y^4 + x^3)\right] dx\,dy \\ &= \int\int_D (12x^5 - 4y^3)\,dx\,dy \\ &= \int_0^1 \left[\int_0^1 (12x^5 - 4y^3)\,dx\right] dy \\ &= \int_0^1 (2 - 4y^3)\,dy = 1. \ ▲\end{aligned}$$

Example 5 Show that if C is the boundary of D, then

$$\int_C PQ\,dx + PQ\,dy = \int\int_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dx\,dy.$$

Solution By Green's theorem,

$$\begin{aligned}\int_C PQ\,dx + PQ\,dy &= \int\int_D \left[\frac{\partial}{\partial x}(PQ) - \frac{\partial}{\partial y}(PQ)\right] dx\,dy \\ &= \int\int_D \left(\frac{\partial P}{\partial x} Q + P\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} Q - P\frac{\partial Q}{\partial y}\right) dx\,dy \\ &= \int\int_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dx\,dy. \ ▲\end{aligned}$$

We can use Green's theorem to obtain a formula for the *area of a region bounded by a curve C.*

Corollary: Area of a Region

If C is a curve that bounds a region D, then the area of D is

$$A = \frac{1}{2}\int_C x\,dy - y\,dx. \tag{4}$$

The proof is as follows. Let $P = -y$, $Q = x$; then by Green's theorem we have

$$\frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int\int_D\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)dx\,dy = \int\int_D dx\,dy$$

which is the area of D. ■

Example 6 Verify formula (4) in the case where D is the disk $x^2 + y^2 \leqslant r^2$.

Solution The area is πr^2. Formula (4) with $x = r\cos t$, $y = r\sin t$, $0 \leqslant t \leqslant 2\pi$, gives

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(r\cos t)(r\cos t)\,dt - (r\sin t)(-r\sin t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} r^2\,dt = \pi r^2,$$

so formula (4) checks. ▲

Example 7 Use formula (4) to find the area bounded by the ellipse C: $x^2/a^2 + y^2/b^2 = 1$.

Solution Parametrize C by $(a\cos t, b\sin t)$, $0 \leqslant t \leqslant 2\pi$. Then (4) gives

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi. \text{ ▲}$$

Exercises for Section 18.4

1. Check the validity of Green's theorem for the region between the curves $y = x^2$ and $y = x$ between $x = 0$ and $x = 1$, with $P = xy$ and $Q = x$.
2. Verify Green's theorem when D is the disk of radius r, center $(0,0)$, and $P = xy^2$, $Q = -yx^2$.
3. Verify Green's theorem for $P = 2y$, $Q = x$, and D the unit disk $x^2 + y^2 \leqslant 1$.
4. Verify Green's Theorem for $P = \cos(xy^2) - xy^2\sin(xy^2)$, $Q = -2x^2y\sin(xy^2)$, and D the ellipse $x^2/4 + y^2/9 \leqslant 1$. [Hint: do not evaluate the line integral directly.]

Exercises 5–8 refer to Fig. 18.4.6. Show how to decompose each region into subregions, each of which is of types 1 and 2.

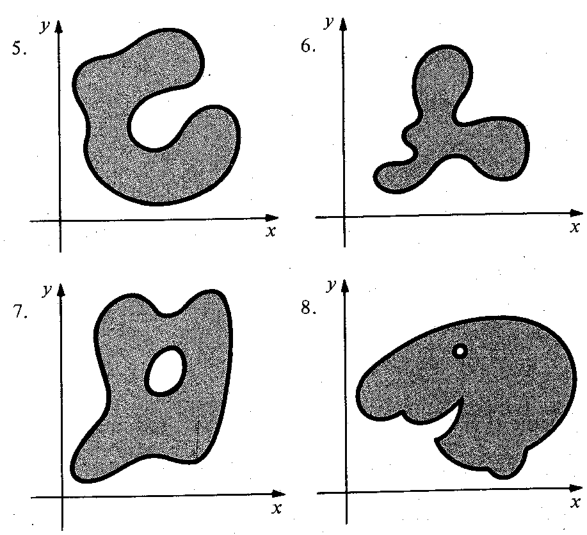

Figure 18.4.6. Subdivide each of these regions into subregions of types 1 and 2.

9. Let C be the ellipse $x^2/a^2 + y^2/b^2 = 1$, and let $\mathbf{\Phi}(x, y) = xy^2\mathbf{i} - yx^2\mathbf{j}$. Write $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ as a double integral using Green's theorem. Evaluate.
10. Let $\mathbf{\Phi}(x, y) = (2y + e^x)\mathbf{i} + (x + \sin(y^2))\mathbf{j}$ and C be the circle $x^2 + y^2 = 1$. Write $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ as a double integral and evaluate.
11. Let C be the boundary of the rectangle $[1,2] \times [1,2]$. Evaluate $\int_C x^2y\,dx + 3yx^2\,dy$ by using Green's theorem.
12. Evaluate $\int_C (x^5 - 2xy^3)\,dx - 3x^2y^2\,dy$, where C is parametrized by (t^8, t^{10}), $0 \leqslant t \leqslant 1$.

Let C be the boundary of the rectangle with sides $x = 1, y = 2, x = 3$, and $y = 3$. Evaluate the integrals in Exercises 13–16.

13. $\int_C (2y^2 + x^5)\,dx + 3y^6\,dy$.
14. $\int_C (xy^2 - y^3)\,dx + (-5x^2 + y^3)\,dy$.
15. $\int_C (3x^4 + 5)\,dx + (y^5 + 3y^2 - 1)\,dy$.
16. $\displaystyle\int_C \left[\frac{2y + \sin x}{1 + x^2}\right] dx + \left[\frac{x + e^y}{1 + y^2}\right] dy$.
17. Suppose that $P\mathbf{i} + Q\mathbf{j}$ is parallel to the tangent vector of a closed curve C.
    (a) Show that $Q\mathbf{i} - P\mathbf{j}$ is perpendicular to the tangent vector.
    (b) Show that $\iint_D (\partial P/\partial x + \partial Q/\partial y)\,dx\,dy = 0$, where D is the region whose boundary is C.

We call $\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ the *Laplacian* of $u = f(x, y)$. Prove the identities in Exercises 18 and 19.

18. $$\iint_D u\,\nabla^2 v\,dx\,dy = -\iint_D \nabla u \cdot \nabla v\,dx\,dy + \int_C u\frac{\partial v}{\partial x}\,dy - u\frac{\partial v}{\partial y}\,dx,$$ (Green's first identity).
19. $$\iint_D (u\,\nabla^2 v - v\,\nabla^2 u)\,dx\,dy = \int_C \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) dy - \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right) dx$$ (Green's second identity).
    [*Hint:* Write down Green's first identity again with u and v interchanged and subtract.]
20. Suppose that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ on D. Show that
$$\int_C \frac{\partial u}{\partial y}\,dx - \frac{\partial u}{\partial x}\,dy = 0.$$

Use formula (4) to determine the area of the regions in the plane bounded by the figures in Exercises 21 and 22.

21. The triangle with vertices $(1,0)$, $(3,4)$, and $(5, -1)$.
22. The rhombus with vertices $(0, -1)$, $(3,0)$, $(1,2)$, and $(4,3)$.
23. Show that the area enclosed by the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \leqslant \theta \leqslant 2\pi$ is $\frac{3}{8}\pi a^2$. (Use Green's theorem.)
24. Find the area bounded by one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $a > 0$, $0 \leqslant \theta \leqslant 2\pi$.
25. Find the area between the curves $y = x^3$ and $y = \sqrt{x}$ by using Green's theorem.
26. Use formula (4) to recover the formula $A = \frac{1}{2}\int_a^b r^2\,d\theta$ for a region in polar coordinates (see Section 10.5).
27. Sketch the proof of Green's theorem for the region shown in Fig. 18.4.7.

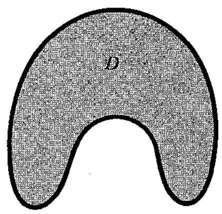

Figure 18.4.7. Prove Green's theorem for this region.

28. Find the work which is done by the force field $(3x + 4y)\mathbf{i} + (8x + 9y)\mathbf{j}$ on a particle which moves once around the ellipse $4x^2 + 9y^2 = 36$ by (a) directly evaluating the line integral and by (b) using Green's theorem.
29. Let $P = y$ and $Q = x$. What is $\int_C P\,dx + Q\,dy$ if C is a closed curve?
30. Use Green's first identity (Exercise 18) with $v = u$ to prove that if $\nabla^2 u = 0$ on D and $u = 0$ on C, then $\nabla u = \mathbf{0}$ on D, and hence $u = 0$ on D.

★31. Green's theorem can be used to give another proof that a differential form $P\,dx + Q\,dy$ defined on the plane is exact if $\partial P/\partial y = \partial Q/\partial x$. (In fact, the argument in Section 18.3 or the one outlined here also works in other regions, such as a disk.) Define f by
$$f(x, y) = \int_0^x a(t, 0)\,dt + \int_0^y b(x, t)\,dt,$$
where $P = a(x, y)$ and $Q = b(x, y)$ and set $u = f(x, y)$. Show that $\partial u/\partial y = Q$. The function f is the line integral of $P\,dx + Q\,dy$ along a horizontal segment C_1 and a vertical segment C_2. Define another function $\hat{u} = \hat{f}(x, y)$ by letting $\hat{C}$ be the path which consists of the vertical segment $\hat{C}_1$ from $(0,0)$ to $(0, y)$, followed by the horizontal segment $\hat{C}_2$ from $(0, y)$ to (x, y) and setting $\hat{f}(x, y) = \int_{\hat{C}_1 + \hat{C}_2} P\,dx + Q\,dy$. Show that $\partial\hat{u}/\partial x = P$. Apply Green's theorem to the rectangle D bounded by $C_1 + C_2 + (-\hat{C}_2) + (-\hat{C}_1)$ to get
$$\iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$$

$$= \int_{C_1 + C_2 + (-\hat{C}_2) + (-\hat{C}_1)} (P\,dx + Q\,dy)$$
$$= \int_{C_1 + C_2} (P\,dx + Q\,dy) - \int_{\hat{C}_1 + \hat{C}_2} (P\,dx + Q\,dy)$$
$$= u - \hat{u};$$

but the double integral over D is zero, since $\partial P/\partial x = \partial Q/\partial y$ by assumption. Conclude that $u = \hat{u}$, to complete the proof.

★32. If $P\,dx + Q\,dy$ satisfies $\partial P/\partial y = \partial Q/\partial x$, use Green's theorem to show that $P\,dx + Q\,dy$ is conservative by showing its integral around "every" closed curve in zero.

★33. *Project:* The formula $A = \frac{1}{2}\int_C x\,dy - y\,dx$ is the basis for the operation of the *planimeter*, a mechanical device for measuring areas. Find out about planimeters from an encyclopedia or the American Mathematical Monthly, Vol. **88**, No. 9, November (1981), p. 701, and relate their operation to this formula for area.

★34. (a) If D is a region to which Green's theorem applies, write the identity of Example 5 this way:

$$\int\int_D \left(P\frac{\partial Q}{\partial x} - P\frac{\partial Q}{\partial y} \right) dx\,dy$$
$$= \int_C PQ\,dx + PQ\,dy$$
$$- \int\int_D \left(Q\frac{\partial P}{\partial x} - Q\frac{\partial P}{\partial y} \right) dx\,dy.$$

How is this like integration by parts?
(b) Elaborate the following statement: Green's theorem is "the fundamental theorem of calculus" in the plane since it relates a double integral to an integral around the boundary, just as the fundamental theorem relates an integral over an interval to a sum over the boundary of the interval (that is, the two endpoints).

18.5 Circulation and Stokes' Theorem

The line integral of a vector field around the boundary of a surface in space equals the surface integral of the curl of the vector field.

If $\mathbf{\Phi}$ is a vector field defined at points in the plane, we can write $\mathbf{\Phi} = P\mathbf{i} + Q\mathbf{j}$. The line integral of $\mathbf{\Phi}$ around a curve C, namely

$$\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy,$$

occurs on the left-hand side of the equation in the statement of Green's theorem. The expression "*circulation* of $\mathbf{\Phi}$ around C" is often used for the number $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$. This terminology arose through the application of Green's theorem to fluid mechanics; we shall now briefly discuss this application.

Imagine a fluid moving in the plane. Each particle of the fluid (or piece of dust suspended in the fluid) has a well-defined velocity. If, at a particular time, we assign to each point (x, y) of the plane the velocity $\mathbf{V}(x, y)$ of the fluid particle moving through (x, y) at that time, we obtain a vector field $\mathbf{V}$ on the plane. See Fig. 18.5.1. The integral $\int_C \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r}$ of $\mathbf{V}$ around a closed curve

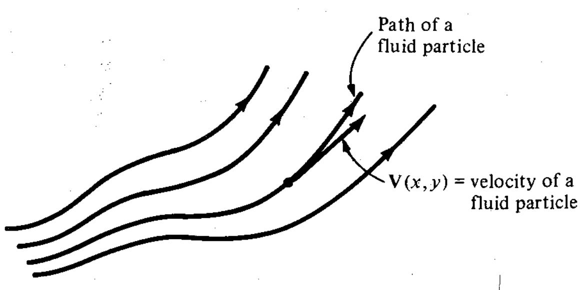

Figure 18.5.1. The velocity field of a fluid.

C represents, intuitively, the sum of the tangential components of $\mathbf{V}$ around C. Thus, if C is traversed counterclockwise, and $\int_C \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r} > 0$, there is a net counterclockwise motion of the fluid. Likewise, if $\int_C \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r} < 0$, the fluid is

circulating clockwise. This explains the origin of the term "circulation" and is illustrated in Fig. 18.5.2.

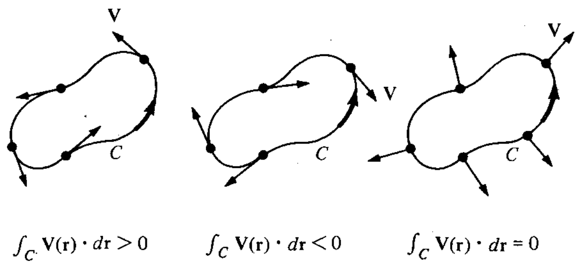

Figure 18.5.2. The intuitive meaning of the possible signs of $\int_C \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r}$.

In Section 18.1, we interpreted the line integral of a force vector field along a curve as the work done by the force on a particle traversing the curve. Notice that the single mathematical concept of a line integral is subject to different interpretations, depending on what physical quantity is represented by the vector field.

The integrand on the right-hand side of Green's theorem,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

is important because, when integrated over the region whose boundary is C, it produces the circulation of $\mathbf{\Phi}$ around C according to Green's theorem.

The Scalar Curl

If $\mathbf{\Phi} = P\mathbf{i} + Q\mathbf{j}$ is a vector field in the plane,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

is called the *scalar curl* of $\mathbf{\Phi}$.

Example 1 In the plane, the vector field

$$\mathbf{V}(x, y) = \frac{y\mathbf{i}}{x^2 + y^2} - \frac{x\mathbf{j}}{x^2 + y^2}$$

approximates (the horizontal part of) the velocity field of water flowing down a drain (see Fig. 18.5.3). (a) Calculate its scalar curl. (b) Is Green's theorem valid for this vector field on the unit disk D, defined by $x^2 + y^2 \leqslant 1$?

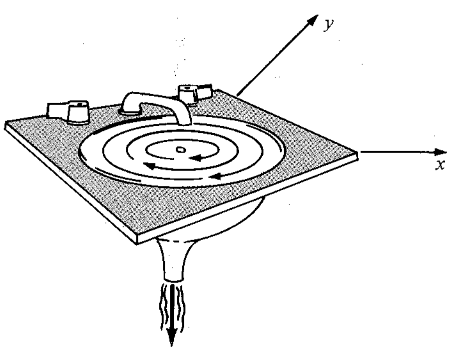

Figure 18.5.3. The velocity field near a drain.

Solution Here $P(x, y) = y/(x^2 + y^2)$ and $Q(x, y) = -x/(x^2 + y^2)$, so the scalar curl is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$
$$= \frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2} + \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = 0.$$

The circulation of $\mathbf{V}$ about the circle $x^2 + y^2 = 1$ is

$$\int_C P\,dx + Q\,dy = \int_C y\,dx - x\,dy = 2 \cdot (\text{area of disk}) = -2\pi.$$

(See formula (4), Section 18.4.) This is an apparent contradiction to Green's theorem! The explanation is the fact that the hypotheses of Green's theorem are not satisfied; $\mathbf{V}$ is not defined at $(0, 0)$. ▲

If we write (scalar curl $\mathbf{\Phi}$) $= \partial Q/\partial x - \partial P/\partial y$, then Green's theorem becomes

$$\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int\int_D (\text{scalar curl } \mathbf{\Phi})\,dx\,dy,$$

where C is the boundary of D. Now choose a point P_0 in the plane and let D_ε be the disk of radius ε about P_0 and C_ε the circle of radius ε. By the mean value theorem,

$$\int\int_{D_\varepsilon} (\text{scalar curl } \mathbf{\Phi})\,dx\,dy = \left[\text{scalar curl } \mathbf{\Phi}(P_\varepsilon)\right]\left[\text{area } D_\varepsilon\right]$$

for some point P_ε in D_ε. Dividing by (area D_ε) and letting $\varepsilon \to 0$ gives

$$(\text{scalar curl } \mathbf{\Phi})(P_0) = \lim_{\varepsilon \to 0}\left[\frac{1}{\text{area } D_\varepsilon}\int_{C_\varepsilon} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}\right],$$

i.e., *the scalar curl may be thought of as the circulation per unit area.*

A fluid moving in space is represented by a vector field $\mathbf{\Phi}(x, y, z)$ in three variables. The generalization of Green's theorem to this case is called *Stokes' theorem*. We shall next prove a special case of this theorem.

Consider a region D in the xy plane and a function f defined in D. The equation $z = f(x, y)$ defines a surface over D. If we refer to Section 15.2, we see that a normal vector to this surface is given by $-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$, and so a unit normal is

$$\mathbf{n} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

The area element on the surface is given by

$$dA = \sqrt{1 + f_x^2 + f_y^2}\;dx\,dy$$

as was shown in Section 17.3. Thus $\mathbf{n}\,dA = (-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k})\,dx\,dy$.

The Surface Integral

If $\mathbf{\Phi} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in space and S is the surface $z = f(x, y)$, the *surface integral* of $\mathbf{\Phi}$ over S is the integral of the normal component of $\mathbf{\Phi}$ over S:

$$\int\int_S \mathbf{\Phi} \cdot \mathbf{n}\,dA = \int\int_D (-Pf_x - Qf_y + R)\,dx\,dy. \tag{1}$$

We shall discuss a physical interpretation of this definition in Section 18.6.

Example 2 Let $\mathbf{\Phi} = x^2\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{\Phi}\cdot\mathbf{n}\,dA$, where S is the graph of the function $z = x + y + 1$ over the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

Solution By formula (1), with $P(x,y,z) = x^2$, $Q(x,y,z) = y^2$, $R(x,y,z) = z$, and $z = f(x,y) = x + y + 1$,

$$\int\int_S \mathbf{\Phi}\cdot\mathbf{n}\,dA = \int\int_D \left[-x^2\cdot 1 - y^2\cdot 1 + (x+y+1)\right]dx\,dy$$

$$= \int_0^1\int_0^1 (x + y + 1 - x^2 - y^2)\,dx\,dy$$

$$= \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \ \blacktriangle$$

Stokes' theorem will involve the concept of *curl* defined as follows.

The Curl of a Vector Field

Let $\mathbf{\Phi} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in space. Its *curl* is defined by

$$\operatorname{curl}\mathbf{\Phi} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

If $\mathbf{\Phi}$ is a vector field in the plane, but regarded as being in space with $R = 0$ and P, Q independent of z, then the curl of $\mathbf{\Phi}$ is just the scalar curl of $\mathbf{\Phi}$ times $\mathbf{k}$.

Formally, we can write $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$, treating it as if it were a vector; then

$$\operatorname{curl}\mathbf{\Phi} = \nabla\times\mathbf{\Phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

which helps one remember the formula.

Example 3 Find the curl of $xy\mathbf{i} - \sin z\,\mathbf{j} + \mathbf{k}$.

Solution

$$\operatorname{curl}\mathbf{\Phi} = \nabla\times\mathbf{\Phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin z & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xy & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & -\sin z \end{vmatrix}\mathbf{k}$$

$$= \cos z\,\mathbf{i} - x\mathbf{k}. \ \blacktriangle$$

Example 4 If f is a twice differentiable function in space, prove that $\nabla\times(\nabla f) = \mathbf{0}$.

Solution Let us write out the components. Since $\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$, we have

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial z\, \partial y} \right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\, \partial x} - \frac{\partial^2 f}{\partial x\, \partial z} \right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial y\, \partial x} \right)\mathbf{k}.$$

Each component is zero because of the symmetry property of mixed partial derivatives. ▲

Now we are ready to state Stokes' theorem. Like Green's theorem, it relates an integral over a surface with an integral around a curve.

Stokes' Theorem

Let D be a region in the plane (to which Green's theorem applies) and S the surface $z = f(x, y)$, where f is twice continuously differentiable. Let ∂D be the boundary of D traversed counterclockwise and ∂S the corresponding boundary of S (see Fig. 18.5.4). If $\mathbf{\Phi}$ is a continuously differentiable vector field in space, then

$$\int_{\partial S} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int\!\!\int_S (\nabla \times \mathbf{\Phi}) \cdot \mathbf{n}\, dA.$$

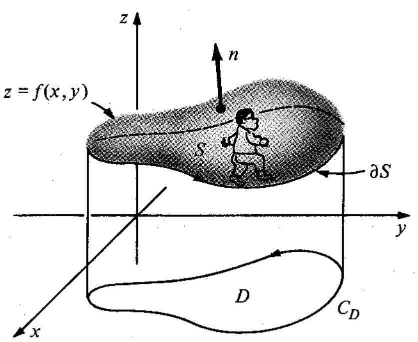

Figure 18.5.4. As you traverse ∂S counterclockwise, the surface is on your left.

Proof of Stokes Theorem Let $\mathbf{\Phi} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, so that

$$\nabla \times \mathbf{\Phi} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

We may use formula (1) to write

$$\int\!\!\int_S \operatorname{curl} \mathbf{\Phi} \cdot \mathbf{n}\, d\mathbf{A} = \int\!\!\int_D \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\left(-\frac{\partial z}{\partial x} \right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\left(-\frac{\partial z}{\partial y} \right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA. \quad (2)$$

On the other hand, if $\boldsymbol{\sigma}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ parametrizes ∂D, then $\boldsymbol{\eta}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + f(x(t), y(t))\mathbf{k}$ is an orientation-preserving parameterization of the oriented simple closed curve ∂S. Thus

$$\int_{\partial S} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt; \quad (3)$$

but, by the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Substituting this expression into (3), we obtain

$$\int_{\partial S} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt$$

$$= \int_{\partial D} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy. \tag{4}$$

Applying Green's theorem to (4) yields

$$\int\int_D \left[\frac{\partial(Q + R\,\partial z/\partial y)}{\partial x} - \frac{\partial(P + R\,\partial z/\partial x)}{\partial y} \right] dA.$$

Now we use the chain rule, remembering that P, Q, and R are functions of x, y, and z, and z is a function of x and y, to obtain

$$\int_{\partial S} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$$

$$= \int\int_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \cdot \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + R \cdot \frac{\partial^2 z}{\partial x\,\partial y} \right) \right.$$

$$\left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \cdot \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\,\partial x} \right) \right] dA.$$

The last two terms in each set of parentheses cancel each other, and we can rearrange terms to obtain the integral (2). ■

As with Green's theorem, Stokes' theorem is valid for a much wider class of surfaces than graphs, but for simplicity we have treated only this case.

Example 5 Let $\mathbf{\Phi} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Show that the integral of $\mathbf{\Phi}$ around an oriented simple closed curve C that is the boundary of a surface S is 0. (Assume S to be the graph of a function.)

Solution By Stokes' theorem,

$$\int_{\partial S} \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{\Phi}) \cdot \mathbf{n}\, dA.$$

However,

$$\nabla \times \mathbf{\Phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0},$$

so $\int_C \mathbf{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int\int_S \nabla \times \mathbf{\Phi} \cdot \mathbf{n}\, dA$
$= \int\int_S \mathbf{0} \cdot \mathbf{n}\, dA = 0.$ ▲

Example 6 Find the integral of $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$ around the triangle with vertices (0, 0, 0), (0, 2, 0) and (0, 0, 2), using Stokes' theorem.

Solution Refer to Fig. 18.5.5. C is the triangle in question and S is a surface it bounds. By Stokes' theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA.$$

Now

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & -z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

Therefore the integral of $\mathbf{F}$ around C is zero. ▲

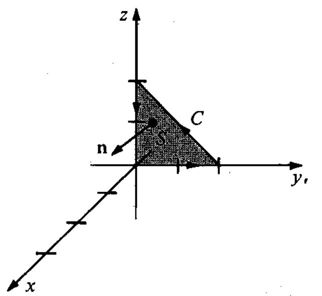

Figure 18.5.5. The curve C of integration for Example 6.

Example 7 Evaluate the integral $\int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA$, where S is the portion of the surface of a sphere defined by $x^2+y^2+z^2=1$ and $x+y+z\geqslant 1$, and where $\mathbf{F} = \mathbf{r}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Solution The surface S and its boundary ∂S are shown in Fig. 18.5.6. We choose the orientation shown.

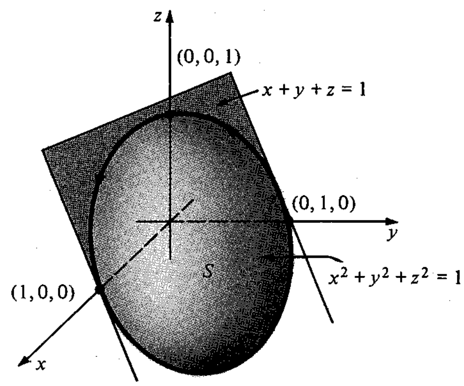

Figure 18.5.6. The surface S in Example 7.

Method 1. Using Stokes' theorem directly. We need to parametrize the boundary circle ∂S, which consists of all unit vectors $\mathbf{v} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ satisfying the equation $x+y+z=1$. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ point from the origin to points on ∂S, and $\mathbf{m} = \frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ points from the origin to the center of the circle ∂S. The radius of the circle ∂S is $\|\frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})-\mathbf{i}\| = (\frac{4}{9}+\frac{1}{9}+\frac{1}{9})^{1/2} = \frac{1}{3}\sqrt{6} = \sqrt{2/3}$. To describe the general point on ∂S, we choose orthogonal unit vectors parallel to the plane $x+y+z=1$, say $\mathbf{u} = (1/\sqrt{2})(\mathbf{i}-\mathbf{j})$ and $\mathbf{v} = (1/\sqrt{6})(\mathbf{i}+\mathbf{j}-2\mathbf{k})$. The general point on ∂S is $\mathbf{m}+\sqrt{2/3}\,[(\cos t)\mathbf{u}+(\sin t)\mathbf{v}]$. As t goes from 0 to 2π, the circle is traversed once. The orientation is correct if the triple product $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{m}$ is positive[3] (Figure 18.5.7). Up to positive factors, this triple product equals the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 3 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 3\cdot 2 = 6.$$

[3] Our original computation of $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{m}$ when writing this solution came out negative, so we interchanged $\mathbf{u}$ and $\mathbf{v}$.

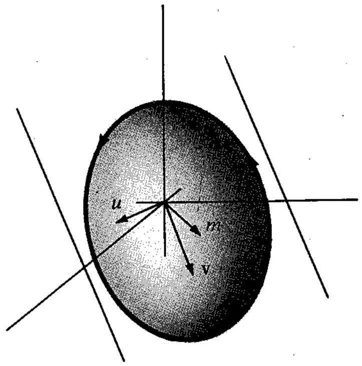

Figure 18.5.7. $\mathbf{u}, \mathbf{v}, \mathbf{m}$ are oriented so that rotations about $\mathbf{m}$ counterclockwise in the $\mathbf{uv}$ plane correspond to the correct orientation for Stokes' theorem.

Now $\mathbf{r} = \mathbf{m} + \sqrt{2/3}\,(\cos t\,\mathbf{u} + \sin t\,\mathbf{v})$ and $d\mathbf{r} = \sqrt{2/3}\,(-\sin t\,\mathbf{u} + \cos t\,\mathbf{v})\,dt$, so by Stokes' theorem,

$$\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dA$$
$$= \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial S} [\mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})] \cdot d\mathbf{r} = \int_{\partial S} (d\mathbf{r} \times \mathbf{r}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= \int_0^{2\pi} \left[\left(\sqrt{\frac{2}{3}}\,(-\sin t\mathbf{u} + \cos t\mathbf{v}) \right) \times \left(\mathbf{m} + \sqrt{\frac{2}{3}}\,(\cos t\,\mathbf{u} + \sin t\,\mathbf{v}) \right) \right] \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt.$$

Now $\mathbf{i} + \mathbf{j} + \mathbf{k} = 3\mathbf{m}$, so the term involving $\mathbf{m}$ drops out, and the integral becomes

$$\frac{2}{3} \int_0^{2\pi} -(\sin^2 t + \cos^2 t)(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = -\frac{2}{3} \int_0^{2\pi} \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} \cdot 6\,dt$$
$$= -\frac{4\pi}{\sqrt{3}}.$$

Method 2. Simplifying the Surface. We compute $\nabla \times \mathbf{F} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. By Stokes' theorem,

$$\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\!\!\int_P (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dA,$$

where P is *any* surface having ∂S as its boundary. We take for P the portion of the plane $x + y + z = 1$ inside the circle in Fig. 18.5.6; $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ is a unit vector orthogonal to the plane, so $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ is constant and equal to $-6/\sqrt{3}$. In method 1, we found the radius of P to be $\sqrt{2/3}$. Thus

$$\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dA = \int\!\!\int_P -\frac{6}{\sqrt{3}}\,dA = \frac{6}{\sqrt{3}}\,(\text{area of } P)$$
$$= -\frac{6}{\sqrt{3}} \cdot \pi \cdot \left(\frac{2}{3}\right) = -\frac{4\pi}{\sqrt{3}}.$$

which agrees with the answer in Method 1. ▲

Just as with the scalar curl, we can show that $\mathbf{n} \cdot \operatorname{curl} \boldsymbol{\Phi}(P_0)$ *is the circulation per unit area at P_0 in the plane through P orthogonal to* $\mathbf{n}$.

Indeed, let D_ε be the disk centered at P_0 with radius ε and lying in the

plane orthogonal to $\mathbf{n}$, and let ∂D_ε be its boundary. See Fig. 18.5.8. By Stokes' theorem,

$$\int\int_{D_\varepsilon}(\nabla\times\mathbf{\Phi})\cdot\mathbf{n}\,dA=\int_{\partial D_\varepsilon}\mathbf{\Phi}\cdot d\mathbf{r}.$$

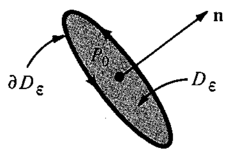

Figure 18.5.8. The curl gives the circulation per unit area.

In one variable calculus we have seen that a mean value theorem holds for integrals (see p. 435). There is a similar result for double and triple integrals; thus there is a point P_ε in D_ε such that

$$\int\int_{D_\varepsilon}(\nabla\times\mathbf{\Phi})\cdot\mathbf{n}\,dA=\left[(\nabla\times\mathbf{\Phi})(P_\varepsilon)\cdot\mathbf{n}\right](\text{area }D_\varepsilon).$$

Thus

$$\mathbf{n}\cdot(\nabla\times\mathbf{\Phi})(P_\varepsilon)=\frac{1}{\pi\varepsilon^2}\int_{\partial D_\varepsilon}\mathbf{\Phi}\cdot d\mathbf{r},\quad\text{and so}\quad \mathbf{n}\cdot(\nabla\times\mathbf{\Phi})(P_0)=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_{\partial D_\varepsilon}\mathbf{\Phi}\cdot d\mathbf{r},$$

as we wanted to show.

Example 8 Let $\mathbf{E}$ and $\mathbf{H}$ be time-dependent electric and magnetic fields in space. Let S be a surface with boundary C. We define

$$\int_C\mathbf{E}\cdot d\mathbf{r}=\text{voltage drop around }C,$$

$$\int\int_S\mathbf{H}\cdot\mathbf{n}\,dA=\text{magnetic flux across }S.$$

Faraday's law (see Fig. 18.5.9) states that the voltage around C equals the negative rate of change of magnetic flux through S.

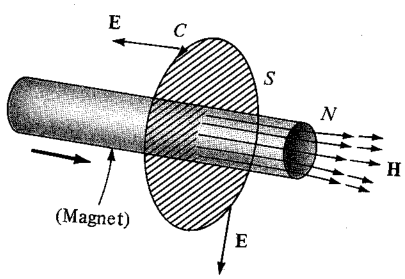

Figure 18.5.9. Faraday's law.

Show that Faraday's law follows from the following differential equation (one of the Maxwell equations):

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{H}}{\partial t}.$$

Solution In symbols, Faraday's law states $\int_C\mathbf{E}\cdot d\mathbf{r}=-(\partial/\partial t)\int\int_S\mathbf{H}\cdot\mathbf{n}\,dA$. By Stokes' theorem, $\int_C\mathbf{E}\cdot d\mathbf{r}=\int\int_S(\nabla\times\mathbf{E})\cdot\mathbf{n}\,dA$. Assuming that we can move $\partial/\partial t$ under the integral sign (see Review Exercise 48, Chapter 17), we get

$$-\frac{\partial}{\partial t}\int\int_S\mathbf{H}\cdot\mathbf{n}\,dA=\int\int_S-\frac{\partial\mathbf{H}}{\partial t}\cdot\mathbf{n}\,dA.$$

Since the two integrals

$$\int\int_S(-(\partial\mathbf{H}/\partial t)\cdot\mathbf{n})\,dA\quad\text{and}\quad\int\int_S(\nabla\times\mathbf{E})\cdot\mathbf{n}\,dA$$

are equal for all S, it must be the case that $\nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t$ (compare the proof of the theorem following Example 3 in Section 18.2). ▲

Exercises for Section 18.5

Calculate the scalar curl of the plane vector fields in Exercises 1–4.

1. $\mathbf{V}(x, y) = y\mathbf{i} - x\mathbf{j}$.
2. $\mathbf{V}(x, y) = xy\mathbf{i} - e^{xy}\mathbf{j}$.
3. $\mathbf{V}(x, y) = \dfrac{x\mathbf{i}}{x^2+y^2} - \dfrac{y\mathbf{j}}{x^2+y^2}$.
4. $\mathbf{V}(x, y) = e^{xy}\mathbf{i} - \dfrac{x}{x^2+y^2}\mathbf{j}$.

Evaluate the surface integrals of the vector fields over the surfaces given in Exercises 5–8.

5. $\mathbf{\Phi} = 3x^2\mathbf{i} - 2yx\mathbf{j} + 8\mathbf{k}$; S is the graph of $z = 2x - y$ over the rectangle $[0,2] \times [0,2]$.
6. $\mathbf{\Phi} = x\mathbf{i} - 2y\mathbf{j} + xz\mathbf{k}$, S is the graph of $z = -x - y - 1$ over the rectangle $[0,1] \times [0,1]$.
7. $\mathbf{\Phi} = x\mathbf{k}$, S is the disk $x^2 + y^2 \leqslant 1$ in the xy plane.
8. $\mathbf{\Phi} = \mathbf{j}$, S is the disk $x^2 + z^2 \leqslant 1$ in the xz plane.

Calculate the curl of the vector fields in Exercises 9–12.

9. $\mathbf{F}(x, y, z) = e^z\mathbf{i} - \cos(xy)\mathbf{j} + z^3y\mathbf{k}$.
10. $\mathbf{\Phi}(x, y, z) = xz\cos x\mathbf{i} - yz\sin x\mathbf{j} - xy\tan y\mathbf{k}$.
11. $$\mathbf{\Phi}(x, y, z) = \frac{yz}{x^2+y^2+z^2}\mathbf{i} - \frac{xz}{x^2+y^2+z^2}\mathbf{j} + \frac{xy}{x^2+y^2+z^2}\mathbf{k}.$$
12. $\mathbf{F}(x, y, z) = (\nabla \times \mathbf{\Phi})(x, y, z)$, where $\mathbf{\Phi}$ is given in Exercise 10.

13. Prove the identity $\operatorname{curl}(f\mathbf{\Phi}) = f\operatorname{curl}\mathbf{\Phi} + \nabla f \times \mathbf{\Phi}$.
14. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, prove that $\nabla \times \mathbf{r} = \mathbf{0}$.
15. Show that Stokes' theorem reduces to $0 = 0$ for $\mathbf{\Phi} = \nabla f$, by evaluating each side directly.
16. Prove the identity
$$\int\int_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, dA = \int_{\partial S} f\nabla g \cdot d\mathbf{r} = -\int_{\partial S} g\nabla f \cdot d\mathbf{r}.$$
17. Let
$$\mathbf{\Phi} = \frac{\mathbf{i}}{y+z} - \frac{x\mathbf{j}}{(y+z)^2} - \frac{x\mathbf{k}}{(y+z)^2}.$$
Show that the integral of $\mathbf{\Phi}$ around an oriented simple curve C that is the boundary of a surface S is zero.
18. Let $\mathbf{\Phi} = (yze^x + xyze^x)\mathbf{i} + xze^x\mathbf{j} + xye^x\mathbf{k}$. Repeat Exercise 17.
19. Let $\mathbf{\Phi} = 2x\mathbf{i} - y\mathbf{j} + (x+z)\mathbf{k}$. Evaluate the integral of $\mathbf{\Phi}$ around the curve consisting of straight lines joining $(1,0,1)$, $(0,1,0)$, and $(0,0,1)$, using Stokes' theorem.
20. Let C consist of straight lines joining $(2,0,0)$, $(0,1,0)$, and $(0,0,3)$. Evaluate the integral of $\mathbf{\Phi}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ around C by using Stokes' theorem.
21. Let $\mathbf{\Phi}$ be perpendicular to the tangent vector of the boundary ∂S of a surface S. Show that $\int\int_S (\nabla \times \mathbf{\Phi}) \cdot \mathbf{n}\, dA = 0$.
22. Let $\mathbf{\Phi} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$ and let C be a curve in a plane with normal $\mathbf{n}$ and enclosing the area A. Find an expression for $\int_C \mathbf{\Phi} \cdot d\mathbf{r}$ using Stokes' theorem.
23. Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ defined by $x + y \geqslant 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j})$.
24. Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$, where S is the portion of the surface of a sphere $x^2 + y^2 + z^2 = 4$ and $3x + 2y - z \geqslant 1$, and where $\mathbf{F}$ is the vector field $\mathbf{r} \times (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.
25. *Ampere's law* states that if the electric current density is described by a vector field $\mathbf{J}$ and the induced magnetic field is $\mathbf{H}$, then the circulation of $\mathbf{H}$ around the boundary C of a surface S equals the integral of $\mathbf{J}$ over S (i.e., the total current crossing S). See Fig. 18.5.10. Show that this is implied by the steady-state *Maxwell equation* $\nabla \times \mathbf{H} = \mathbf{J}$.

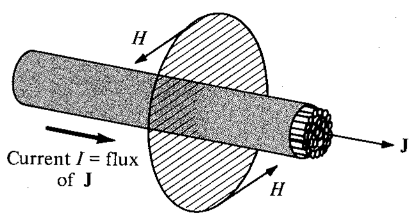

Figure 18.5.10. Ampere's law.

26. Let $\mathbf{F} = x^2\mathbf{i} + (2xy + x)\mathbf{j} + z\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ oriented counterclockwise and S the disk $x^2 + y^2 \leqslant 1$. Determine:
    (a) The integral of $\mathbf{F}$ over S.
    (b) The circulation of $\mathbf{F}$ around C.
    (c) Find the integral of $\nabla \times \mathbf{F}$ over S. Verify Stokes' theorem directly in this case.

★27. Imagine a fluid moving in space. Take a paddle wheel on the end of a stick and put it in the fluid. Move the stick around until the paddle wheel rotates the fastest in a counter-clockwise direction. Your stick now points in the direction of the curl. Justify.

★28. Generalize Example 7 by replacing $x + y + z \geqslant 1$ by $ax + by + cz \geqslant d$ and $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ by $\mathbf{F} = \mathbf{r} \times (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$. (What inequality is needed to ensure that the plane $ax + by + cz = d$ intersects the sphere $x^2 + y^2 + z^2 = 1$?)

★29. Prove that Stokes' theorem holds for the surface $x^2 + y^2 + z^2 = 1$, $z \geqslant -1/2$.

18.6 Flux and the Divergence Theorem

The integral of the normal component of a vector field over a closed surface equals the integral of its divergence over the enclosed volume.

Let $\mathbf{V}$ be the velocity field of a fluid moving in the plane. In the previous section, we explained why the line integral of $\mathbf{V}$ around a closed curve C is called the circulation of $\mathbf{V}$ around C. The line integral is the integral of the tangential component of $\mathbf{V}$. The integral around C of the normal component of $\mathbf{V}$ also has physical meaning.

Imagine first that $\mathbf{V}$ is constant and C is a line segment; see Fig. 18.6.1.

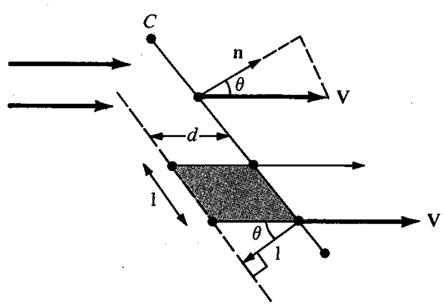

Figure 18.6.1. The amount of fluid crossing C per unit time is the normal component of $\mathbf{V}$ times the length of C.

Suppose we consider a parallelogram consisting of a unit area of fluid, the shaded area in Fig. 18.6.1. The parallelogram's base is one unit in length along C and has its other side parallel to $\mathbf{V}$. Since its area is one, the other side has length $d = 1/\cos\theta$, where θ is the angle between $\mathbf{n}$ and $\mathbf{V}$. It takes this parallelogram $t = d/\|\mathbf{V}\| = 1/[\cos\theta\|\mathbf{V}\|]$ units of time to cross C. Thus $\cos\theta\|\mathbf{V}\|$ square units of fluid cross each unit length of C per unit time. Since $\mathbf{n}$ has unit length, this rate equals $\mathbf{V}\cdot\mathbf{n}$.

If we now imagine C to consist of straight line segments and $\mathbf{V}$ to be constant across each one, we are led to interpret the integral of the normal component of $\mathbf{V}$ along C, that is,

$$\int_C \mathbf{V}\cdot\mathbf{n}\,ds,$$

as the amount of fluid crossing C per unit of time. This integral is the *flux* of $\mathbf{V}$ across C.

Let C be parametrized by $\boldsymbol{\sigma}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then a unit tangent vector is

$$\mathbf{t} = \frac{x'\mathbf{i} + y'\mathbf{j}}{\sqrt{(x')^2 + (y')^2}}, \qquad \text{where } x' = dx/dt \text{ and } y' = dy/dt.$$

The element of length is

$$ds = \sqrt{(x')^2 + (y')^2}\,dt$$

and a unit normal is

$$\mathbf{n} = \frac{y'\mathbf{i} - x'\mathbf{j}}{\sqrt{(x')^2 + (y')^2}}.$$

This $\mathbf{n}$ has length 1 and is perpendicular to $\mathbf{t}$, as is easily checked. We chose

this $\mathbf{n}$ and not its negative so that if C is a closed curve traversed counterclockwise, $\mathbf{n}$ will be the unit *outward* normal, as in Fig. 18.6.2. If $\mathbf{V} = P\mathbf{i} + Q\mathbf{j}$, then substitution of the above formulas for $\mathbf{n}$ and ds gives

$$\mathbf{V} \cdot \mathbf{n}\, ds = (Py' - Qx')\, dt.$$

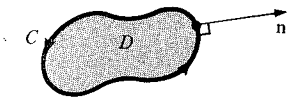

Figure 18.6.2. $\mathbf{n} = \dfrac{y'\mathbf{i} - x'\mathbf{j}}{\sqrt{(x')^2 + (y')^2}}$ is the unit outward normal.

This leads to the following definition.

The Flux of a Vector Field

The *flux* of $\mathbf{V}$ across a curve C is defined to be

$$\int_C \mathbf{V} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx.$$

If C is parametrized by $\boldsymbol{\sigma}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leqslant t \leqslant b$, the flux equals

$$\int_a^b \left(P(x(t), y(t))\frac{dy}{dt} - Q(x(t), y(t))\frac{dx}{dt} \right) dt,$$

as our calculations above show.

The divergence theorem relates the flux of a vector field $\mathbf{V}$ across C to the integral of the *divergence* of $\mathbf{V}$ over D defined as follows.

Divergence in the Plane

The *divergence* of a vector field $\mathbf{V} = P\mathbf{i} + Q\mathbf{j}$ is the function given by

$$\operatorname{div}\mathbf{V} = \boldsymbol{\nabla} \cdot \mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

We can remember the formula for $\operatorname{div}\mathbf{V}$ in the plane by writing the "dot product" of $\boldsymbol{\nabla} = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j}$ with $\mathbf{V}$, where instead of multiplying $\partial\ \partial x$ and P, we let $\partial/\partial x$ *operate on* P. This is similar to the way we regarded curl $\boldsymbol{\Phi} = \boldsymbol{\nabla} \times \boldsymbol{\Phi}$ as the cross product of $\boldsymbol{\nabla} = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$ with $\boldsymbol{\Phi}$ (see Example 3, Section 18.5).

Gauss' Divergence Theorem in the Plane

Let D be a region in the plane to which Green's theorem applies and let C be its boundary traversed in a counterclockwise direction. Then

$$\int_C \mathbf{V} \cdot \mathbf{n}\, ds = \int\int_D (\operatorname{div}\mathbf{V})\, dx\, dy.$$

The proof is as follows. The left-hand side equals

$$\int_C P\, dy - Q\, dx.$$

By Green's theorem, this equals (with P replaced by $-Q$ and Q by P in the statement of Green's theorem)

$$\int\int_D \left(\frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y} \right) dx\, dy = \int\int_D (\operatorname{div}\mathbf{V})\, dx\, dy. \blacksquare$$

Example 1 Calculate the flux of $\mathbf{V} = x\cos y\,\mathbf{i} - \sin y\,\mathbf{j}$ across the boundary of the unit square in the plane with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Solution The divergence of $\mathbf{V}$ is

$$\operatorname{div}\mathbf{V} = \frac{\partial}{\partial x}(x\cos y) + \frac{\partial}{\partial y}(-\sin y) = \cos y - \cos y = 0,$$

so by the divergence theorem, the flux across any closed curve is zero. Thus the flux across the given boundary is zero. (Notice that directly computing the flux across C is possible, but more tedious.) ▲

A vector field in the plane is called *incompressible* or *divergence free* if $\operatorname{div}\mathbf{V} = 0$. This terminology arises from the divergence theorem and the example in which $\mathbf{V}$ is the velocity of a fluid. In fact, the divergence theorem implies that the flux across closed curves is zero, that is, the net area of fluid entering and leaving the region enclosed by C is zero. For a compressible fluid, it can happen that the fluid inside C is squeezed so that the net flux across C is negative. In this case, $\operatorname{div}\mathbf{V}$ would be negative. Likewise, if $\operatorname{div}\mathbf{V} > 0$ in a region, the fluid is expanding. See Fig. 18.6.3.

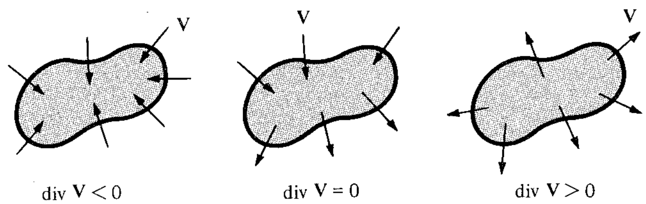

Figure 18.6.3. A compressing, incompressible and expanding fluid.

Example 2 Figure 18.6.4 shows some flow lines for a fluid moving in the plane with velocity field $\mathbf{V}$. What would you guess[4] the sign of $\operatorname{div}\mathbf{V}$ to be at points A, B, C, and D?

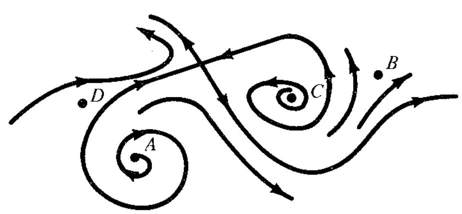

Figure 18.6.4. Find the sign of $\operatorname{div}\mathbf{V}$.

Solution The fluid appears to be emerging from small regions near A, B, and C, so at these points it is reasonable to suppose $\operatorname{div}\mathbf{V} > 0$. At D the fluid appears to be converging, so there $\operatorname{div}\mathbf{V} < 0$. ▲

We saw that a generalization of Green's theorem to three dimensions which relies on the idea of circulation is given by Stokes' theorem. It is natural to also seek a generalization of the divergence theorem to three dimensions.

Let $\mathbf{V}$ be a vector field defined in space. Reasoning as we did in the plane, we see that if $\mathbf{V}$ represents the velocity field of a fluid and S is a surface, then the surface integral $\iint_S \mathbf{V}\cdot\mathbf{n}\,dA$ is the volume of fluid crossing the surface S in the direction of the normal $\mathbf{n}$ per unit time. Thus we call $\iint_S \mathbf{V}\cdot\mathbf{n}\,dA$ the *flux* of $\mathbf{V}$ across S.

If $\mathbf{V} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we can generalize our definition of two-dimensional divergence by setting

$$\operatorname{div}\mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

again called the *divergence* of the three-dimensional vector field $\mathbf{V}$.

[4] One has to guess, because one cannot really tell from the picture without knowing how fast the fluid is moving.

Gauss' Divergence Theorem in Space

Let W be a region in space which is of type I, II, and III (see Section 17.4), and let ∂W be the surface of W with $\mathbf{n}$ the outward pointing unit normal. Let $\mathbf{V}$ be a vector field defined on W. Then

$$\iiint_W (\operatorname{div}\mathbf{V})\,dx\,dy\,dz = \iint_{\partial W}(\mathbf{V}\cdot\mathbf{n})\,dA.$$

In words, the total flux across the boundary of W equals the total divergence in W.

Proof of Gauss' Theorem It is sufficient to prove the three equalities

$$\iint_{\partial W} P\mathbf{i}\cdot\mathbf{n}\,dA = \iiint_W \frac{\partial P}{\partial x}\,dx\,dy\,dz, \tag{1}$$

$$\iint_{\partial W} Q\mathbf{j}\cdot\mathbf{n}\,dA = \iiint_W \frac{\partial Q}{\partial y}\,dx\,dy\,dz, \tag{2}$$

$$\iint_{\partial W} R\mathbf{k}\cdot\mathbf{n}\,dA = \iiint_W \frac{\partial R}{\partial z}\,dx\,dy\,dz. \tag{3}$$

This is because

$$\iiint_W (\operatorname{div}\mathbf{V})\,dx\,dy\,dz = \iiint_W \frac{\partial P}{\partial x}\,dx\,dy\,dz + \iiint_W \frac{\partial Q}{\partial y}\,dy\,dy\,dz + \iiint_W \frac{\partial R}{\partial z}\,dx\,dy\,dz,$$

and

$$\iint_{\partial W}\mathbf{V}\cdot\mathbf{n}\,dA = \iint_{\partial W}(P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{n}\,dA = \iint_{\partial W}P\mathbf{i}\cdot\mathbf{n}\,dA + \iint_{\partial W}Q\mathbf{j}\cdot\mathbf{n}\,dA + \iint_{\partial W}R\mathbf{k}\cdot\mathbf{n}\,dA.$$

The equality (3) will be proved here; the other two are proved in an analogous fashion.

Express W by the inequalities

$$f_2(x,y)\leqslant z\leqslant f_1(x,y),\qquad (x,y)\text{ in } D$$

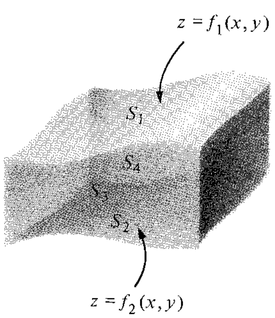

Figure 18.6.5. A region W of type 1. The four sides of ∂W, namely S_3, S_4, S_5, S_6 have normals perpendicular to the z axis.

for functions f_1 and f_2 on a domain D in the xy plane. (See Fig. 18.6.5.) The boundary of W is closed surface whose top S_1 is the graph of $z=f_1(x,y)$, (x,y) in D, and whose bottom S_2 is the graph of $z=f_2(x,y)$, (x,y) in D. The four other sides of ∂W (if they are not reduced to curves) consist of surfaces S_3, S_4, S_5, and S_6 whose normals are always perpendicular to the z axis. We claim that

$$\iiint_W \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint_D\big[R(x,y,f_1(x,y)) - R(x,y,f_2(x,y))\big]\,dx\,dy. \tag{4}$$

Indeed, by the fundamental theorem of calculus and the reduction to iterated integrals,

$$\iiint_W \frac{\partial R}{\partial z}\,dz\,dy\,dx = \iint_D\Big[R(x,y,z)\Big|_{z=f_2(x,y)}^{f_1(x,y)}\Big]\,dy\,dx = \iint_D\big[R(x,y,f_1(x,y)) - R(x,y,f_2(x,y))\big]\,dx\,dy.$$

Next we break up the left side of (3) into the sum of six terms:

$$\iint_{\partial W} R\mathbf{k}\cdot\mathbf{n}\,dA = \iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1\,dA + \iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2\,dA + \sum_{i=3}^{6}\iint_{S_i} R\mathbf{k}\cdot\mathbf{n}_i\,dA. \tag{5}$$

Since on each of S_3, S_4, S_5, and S_6, the normal $\mathbf{n}_i$ is perpendicular to $\mathbf{k}$, we have $\mathbf{k}\cdot\mathbf{n}_i = 0$ along these faces, and so the integral (5) reduces to

$$\iint_{\partial W} R\mathbf{k}\cdot\mathbf{n}\,dA = \iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1\,dA + \iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2\,dA. \tag{6}$$

The surface S_2 is defined by $z = f_2(x, y)$, so

$$\mathbf{n}_2 = \frac{(\partial f_2/\partial x)\mathbf{i} + (\partial f_2/\partial y)\mathbf{j} - \mathbf{k}}{\sqrt{(\partial f_2/\partial x)^2 + (\partial f_2/\partial y)^2 + 1}}\,. \tag{7}$$

Thus

$$\mathbf{n}_2\cdot\mathbf{k} = \frac{-1}{\sqrt{(\partial f_2/\partial x)^2 + (\partial f_2/\partial y)^2 + 1}}\,,$$

and so

$$\iint_{S_2} R(\mathbf{k}\cdot\mathbf{n}_2)\,dA$$

$$= \iint_D R(x, y, f_2(x, y))\left[\frac{-1}{\sqrt{\left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + 1}}\right]\sqrt{\left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + 1}\;dA$$

$$= -\iint_D R(x, y, f_2(x, y))\,dx\,dy. \tag{8}$$

The formula for $\mathbf{n}_1$ is similar to formula (7) for $\mathbf{n}_2$. However, $\mathbf{n}_1$ points upward, so the numerator of $\mathbf{n}_1$ is $-(\partial f_1/\partial x)\mathbf{i} - (\partial f_1/\partial y)\mathbf{j} + \mathbf{k}$ (Note the positive $\mathbf{k}$ component). Thus

$$\mathbf{n}_1\cdot\mathbf{k} = \left[\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + 1\right]^{-1/2},$$

and so

$$\iint_{S_1} R(\mathbf{k}\cdot\mathbf{n}_1)\,dA = \iint_D R(x, y, f_1(x, y))\,dx\,dy. \tag{9}$$

Substituting (8) and (9) in (6) gives

$$\begin{aligned}\iint_{\partial W} & R(\mathbf{k}\cdot\mathbf{n})\,dA \\ &= \iint_D R(x, y, f_1(x, y))\,dx\,dy - \iint_D R(x, y, f_2(x, y))\,dx\,dy \\ &= \iint_D \left[R(x, y, f_1(x, y)) - R(x, y, f_2(x, y))\right]dx\,dy \\ &= \iiint_W \frac{\partial R}{\partial z}\,dx\,dy\,dz, \qquad \text{by formula (4).}\end{aligned}$$

This proves that

$$\iint_{\partial W} R\mathbf{k}\cdot\mathbf{n}\,dA = \iiint_W \frac{\partial R}{\partial z}\,dx\,dy\,dz$$

which is the identity (3) which we wanted to show. ∎

As with Green's and Stokes' theorems, Gauss' theorem holds for regions more general than those of types I, II, and III. This assertion may be established by breaking up a region W into smaller subregions $W_1, \ldots, W_n$ each of which is of types I, II, and III. We apply Gauss' theorem to each of these smaller regions and add the results. The surface integrals along common boundaries interior to W cancel, and we are left with Gauss' theorem for the original region W.

Example 3 Evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA, \qquad \text{where} \quad \mathbf{F}(x,y,z) = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$$

and S is the surface of the cylinder W defined by $x^2 + y^2 \leqslant 1$, $-1 \leqslant z \leqslant 1$.

Solution One can compute the integral directly, but it is easier to use the divergence theorem.

Since S is the boundary of the region W, the divergence theorem gives $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_W \operatorname{div}\mathbf{F}\,dx\,dy\,dz$. Now

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(y)\\ &= x^2 + y^2,\end{aligned}$$

and so

$$\begin{aligned}\iiint_W \operatorname{div}\mathbf{F}\,dx\,dy\,dz &= \iiint_W (x^2+y^2)\,dx\,dy\,dz\\ &= \int_{-1}^{1}\left(\iint_{x^2+y^2\leqslant 1}(x^2+y^2)\,dx\,dy\right)dz\\ &= 2\iint_{x^2+y^2\leqslant 1}(x^2+y^2)\,dx\,dy.\end{aligned}$$

We change variables to polar coordinates to evaluate the double integral:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad 0 \leqslant r \leqslant 1, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

Replacing $x^2 + y^2$ by r^2 and $dx\,dy$ by $r\,dr\,d\theta$, we have

$$\iint_{x^2+y^2\leqslant 1}(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\left(\int_0^1 r^3\,dr\right)d\theta = \frac{1}{2}\pi.$$

Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_W \operatorname{div}\mathbf{F}\,dx\,dy\,dz = \pi. \ \blacktriangle$$

Example 4 Let $\mathbf{V} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Calculate the flux of $\mathbf{V}$ across S.

Solution By the divergence theorem, the flux of $\mathbf{V}$ across S equals

$$\iiint_W \operatorname{div}\mathbf{V}\,dx\,dy\,dz,$$

where W is the ball $x^2 + y^2 + z^2 \leqslant 1$. However,

$$\operatorname{div}\mathbf{V}(x,y,z) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) = 6.$$

Thus the flux is

$$6 \times \text{volume}(W) = 6\cdot\frac{4}{3}\pi = 8\pi. \ \blacktriangle$$

Example 5 Calculate the flux of $\mathbf{V}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ across the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution For $\mathbf{V}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, $\operatorname{div}\mathbf{V} = 3x^2 + 3y^2 + 3z^2$. By Gauss' theorem in space, the flux equals

$$\int\int\int_W (\operatorname{div}\mathbf{V})\, dx\, dy\, dz.$$

Using spherical coordinates (Section 17.5), this becomes

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{12}{5}\pi. \blacktriangle$$

Example 6 Prove the vector identities (a) $\nabla\cdot(f\mathbf{\Phi}) = (\nabla f)\cdot\mathbf{\Phi} + f\nabla\cdot\mathbf{\Phi}$ (b) $\nabla\cdot(\nabla\times\mathbf{\Phi}) = 0$.

Solution Suppose that $\mathbf{\Phi} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

(a)

$$\begin{aligned}\nabla\cdot(f\mathbf{\Phi}) &= \frac{\partial(fa)}{\partial x} + \frac{\partial(fb)}{\partial y} + \frac{\partial(fc)}{\partial z}\\ &= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b + \frac{\partial f}{\partial z}c + f\left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) = (\nabla f)\cdot\mathbf{\Phi} + f\nabla\cdot\mathbf{\Phi}.\end{aligned}$$

(b)

$$\nabla\cdot(\nabla\times\mathbf{\Phi}) = \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right).$$

By commutativity of partial derivatives, all terms cancel to give 0 as the result. ▲

Example 7 A basic law of electrostatics is that an electric field $\mathbf{E}$ in space satisfies $\operatorname{div}\mathbf{E} = \rho$, where ρ is the charge density. Show that the flux of $\mathbf{E}$ across a closed surface equals the total charge *inside* the surface.

Solution Let W be a region in space with boundary surface S. By the divergence theorem,

$$\begin{aligned}\left\{\begin{matrix}\text{flux of } \mathbf{E}\\ \text{across } S\end{matrix}\right\} &= \int\int_S \mathbf{E}\cdot\mathbf{n}\, dA\\ &= \int\int\int_W \operatorname{div}\mathbf{E}\, dx\, dy\, dz\\ &= \int\int\int_W \rho(x, y, z)\, dx\, dy\, dz,\end{aligned}$$

since $\operatorname{div}\mathbf{E} = \rho$ by assumption; but since ρ is the charge per unit volume,

$$Q = \int\int\int_W \rho\, dx\, dy\, dz$$

is the total charge inside S. ▲

Exercises for Section 18.6

Calculate the divergence of the vector fields in Exercises 1–4.

1. $\mathbf{\Phi}(x, y) = x^3\mathbf{i} - x\sin(xy)\mathbf{j}$.
2. $\mathbf{\Phi}(x, y) = y\mathbf{i} - x\mathbf{j}$.
3. $\mathbf{F}(x, y) = \sin(xy)\mathbf{i} - \cos(x^2y)\mathbf{j}$.
4. $\mathbf{F}(x, y) = xe^y\mathbf{i} - [y/(x+y)]\mathbf{j}$.
5. Calculate the flux of $\mathbf{\Phi}(x, y) = x^2\mathbf{i} - y^3\mathbf{j}$ across the perimeter of the square whose vertices are $(-1, -1), (-1, 1), (1, 1), (1, -1)$.
6. Evaluate the flux of $\mathbf{\Phi}(x, y, z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j}$ out of the unit circle $x^2 + y^2 = 1$ in the plane.

7. Calculate the flux of $\mathbf{\Phi}(x, y) = y\mathbf{i} + e^x\mathbf{j}$ across the boundary of the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.
8. Calculate the flux of $x^3\mathbf{i} + y^3\mathbf{j}$ out of the unit circle $x^2 + y^2 = 1$.
9. Fig. 18.6.6 shows some flow lines for a fluid moving in the plane. Let $\mathbf{V}$ be the velocity field. At which of the indicated points A, B, C, D can one reasonably expect that (a) $\operatorname{div} \mathbf{V} > 0$? (b) $\operatorname{div} \mathbf{V} < 0$?

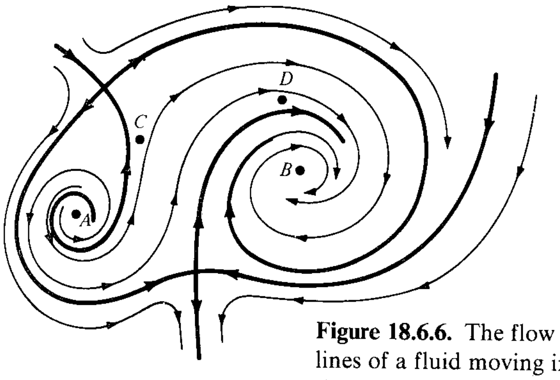

Figure 18.6.6. The flow lines of a fluid moving in the plane.

10. Fig. 18.6.7 shows some flow lines for a fluid moving in the plane. Let $\mathbf{V}$ be the velocity field. At which of the indicated points A, B, C, D can one reasonably expect that (a) $\operatorname{div} \mathbf{V} > 0$? (b) $\operatorname{div} \mathbf{V} < 0$?

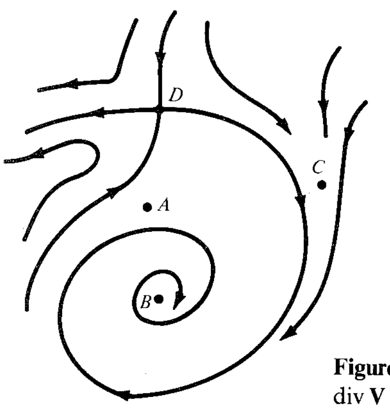

Figure 18.6.7. Where is $\operatorname{div} \mathbf{V} > 0$? < 0?

Find the divergence of the vector fields in Exercises 11–14.

11. $\mathbf{V}(x, y, z) = e^{xy}\mathbf{i} - e^{xy}\mathbf{j} + e^{yz}\mathbf{k}$.
12. $\mathbf{V}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.
13. $\mathbf{V}(x, y, z) = x\mathbf{i} + (y + \cos x)\mathbf{j} + (z + e^{xy})\mathbf{k}$.
14. $\mathbf{V}(x, y, z) = x^2\mathbf{i} + (x + y)^2\mathbf{j} + (x + y + z)^2\mathbf{k}$.
15. Find the flux of $\mathbf{\Phi}(x, y, z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$ out of the unit sphere.
16. Evaluate the flux of $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of the unit sphere.
17. Evaluate $\iint_{\partial W} \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ and W is the unit cube in the first octant. Perform the calculation directly and check by using the divergence theorem.
18. Evaluate the surface integral $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and ∂S is the surface of the cylinder $x^2 + y^2 \leqslant 1$, $0 \leqslant z \leqslant 1$.
19. Suppose a vector field $\mathbf{V}$ is tangent to the boundary of a region W in space. Prove that $\iiint_W (\operatorname{div} \mathbf{V})\, dx\, dy\, dz = 0$.
20. Prove the identity
$$\nabla \cdot (\mathbf{F} \times \mathbf{\Phi}) = \mathbf{\Phi} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{\Phi}).$$
21. Prove that
$$\iiint_W (\nabla f) \cdot \mathbf{\Phi}\, dx\, dy\, dz$$
$$= \iint_{\partial W} f\mathbf{\Phi} \cdot \mathbf{n}\, dA - \iiint_W f \nabla \cdot \mathbf{\Phi}\, dx\, dy\, dz.$$
22. Prove that $(\partial/\partial t)(\nabla \cdot \mathbf{H}) = 0$ from the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{H}/\partial t$ (Example 8, Section 18.5).
23. i(i) Prove that $\nabla \cdot \mathbf{J} = 0$ from the steady state Maxwell equation $\nabla \times \mathbf{H} = \mathbf{J}$ (Exercise 25, Section 18.5).
    (ii) Argue physically that the flux of $\mathbf{J}$ through any closed surface is zero (conservation of charge). Use this to deduce $\nabla \cdot \mathbf{J} = 0$ from Gauss' theorem.
24. (a) Use Gauss' theorem to show that
$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA,$$
    where S_1 and S_2 have a common boundary.
    (b) Prove the same assertion using Stokes' Theorem.
    (c) Where was this used in Example 7, Section 18.5?

★25. Let ρ be a continuous function of $\mathbf{q} = (x, y, z)$ such that $\rho(\mathbf{q}) = 0$ except for $\mathbf{q}$ in some region Ω. The *potential* of ρ is defined as the function
$$\phi(\mathbf{p}) = \iiint_\Omega \frac{\rho(\mathbf{q})}{4\pi\|\mathbf{p} - \mathbf{q}\|}\, dx\, dy\, dz,$$
where $\|\mathbf{p} - \mathbf{q}\|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$.
    (a) Show that for a region D in space $\iint_{\partial D} \nabla\phi \cdot \mathbf{n}\, dA = \iiint_D \rho\, dx\, dy\, dz$.
    (b) Show that ϕ satisfies *Poisson's equation* $\nabla^2\phi = \rho$.

Review Exercises for Chapter 18

Evaluate the line integrals in Exercises 1–10.

1. $\int_C xy\,dx + x\sin y\,dy$, where C is the straight line segment joining $(0,1,1)$ to $(2,2,-3)$.
2. $\int_C x\,dy$, where C is the unit circle $x^2+y^2=1$ traversed counterclockwise.
3. $\int_C xe^y\,dx - ye^x\,dy$ where C is the straight line segment joining $(0,1,0)$ to $(2,1,1)$.
4. $\int_C x^2\,dx + y^2\,dy$, where C is the circle $x^2+y^2=1$, traversed counterclockwise.
5. $\int_C \nabla(xy^2\cos z)\cdot d\mathbf{r}$ where C is a curve in space joining $(0,1,0)$ to $(8,2,\pi)$.
6. $\int_C \nabla(\cos(xyz))\cdot d\mathbf{r}$, where C is a curve in space joining $(0,0,0)$ to $(1,\pi/2,1)$.
7. $\int_C \frac{\partial}{\partial x}(e^{xyz})\,dx + \frac{\partial}{\partial y}(e^{xyz})\,dy + \frac{\partial}{\partial z}(e^{xyz})\,dz$, where C is a curve in space joining $(1,1,1)$ to $(0,2,0)$.
8. $\int_C \frac{\partial}{\partial x}(x\cos yz)\,dx + \frac{\partial}{\partial y}(x\cos yz)\,dy + \frac{\partial}{\partial z}(x\cos yz)\,dz$, where C is a curve in space joining $(0,0,0)$ to $(1,\pi,1)$.
9. $\int_C \sin x\,dx - \ln z\,dy + xy\,dz$, where C is parametrized by $(2t+1,\ln t,t^2)$, $1\leqslant t\leqslant 2$.
10. $\int_C xy^2\,dx + (y+z)\,dy + [\sin(e^y)]\,dz$, where C is parametrized by $(\sin 2t,\cos t,5)$, $0\leqslant t\leqslant 2\pi$.
11. Let $f(x,y)=x^2\sin y - xy^2$. Evaluate the integral $\int_C(\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy$, where C is parametrized by:
    (a) $(\tan\pi t,\ln(t+1))$; $-\frac{1}{4}\leqslant t\leqslant\frac{1}{4}$.
    (b) $(t\cos^2(\pi t/4),e^{1+t})$; $0\leqslant t\leqslant 2$.
    (c) $(t^3-2t^2+1,\sin^5(\pi t/2)-2t^{3/2})$; $0\leqslant t\leqslant 1$.
12. If $f(x,y,z)=e^{x^2+y}-\ln(z^2+1)$, evaluate the line integral $\int_C\nabla f(\mathbf{r})\cdot d\mathbf{r}$ along the curve given by $\boldsymbol{\sigma}(t)=(-t^2,t^3+1,t\sin\frac{15}{2}\pi t)$, $0\leqslant t\leqslant 1$.

In Exercises 13–16, let $f(x,y,z)=xze^y - z^3/(1+y^2)$. Evaluate the line integral $\int_C(\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy + (\partial f/\partial z)\,dz$, where C has the given parametrization.

13. $\boldsymbol{\sigma}(t)=\left(\sqrt{t}\sin\frac{\pi}{4}(1+t),t^2-1,\frac{2-t^2}{2+t^2}\right)$; $0\leqslant t\leqslant 1$.
14. $\boldsymbol{\sigma}(t)=(\cos^4 t,e^{\sin(\pi t/2)},2-t)$; $0\leqslant t\leqslant 1$.
15. $\boldsymbol{\sigma}(t)=\left(\cos\frac{\pi}{2}t,\sin\frac{\pi}{2}t,t\right)$; $-1\leqslant t\leqslant 1$.
16. $\boldsymbol{\sigma}(t)=(e^t,e^{-t},t^2)$, $0\leqslant t\leqslant 1$.
17. Find the work done moving a particle along the path $\boldsymbol{\sigma}(t)=(t,t^2)$, $0\leqslant t\leqslant 1$ subject to the force $\boldsymbol{\Phi}(x,y)=e^x\mathbf{i}-xe^{xy}\mathbf{j}$.
18. Find the work done moving a particle from $(0,0,0)$ to $(1,1,1)$ along the path
$$(x,y,z)=(t^3,t,t^2)$$
subject to the force $\boldsymbol{\Phi}(x,y,z)=z\cos x\mathbf{j}+8yz\mathbf{k}$.

In Exercises 19–22, determine if the given vector field $\boldsymbol{\Phi}$ is conservative; if it is, find an f such that $\boldsymbol{\Phi}=\nabla f$.

19. $\boldsymbol{\Phi}(x,y)=x\tan y\mathbf{i}+x\sec^2 y\mathbf{j}$.
20. $\boldsymbol{\Phi}(x,y)=\tan y\mathbf{i}+x\sec^2 y\mathbf{j}$.
21. $\boldsymbol{\Phi}(x,y)=3x^2y^2\mathbf{i}+2x^3y^2\mathbf{j}$.
22. $\boldsymbol{\Phi}(x,y)=3x^2y^2\mathbf{i}+2x^3y\mathbf{j}$.
23. In Exercise 30, Section 18.2, it was shown that a vector field
$$\boldsymbol{\Phi}(x,y,z)=a(x,y,z)\mathbf{i}+b(x,y,z)\mathbf{j}+c(x,y,z)\mathbf{k}$$
defined in all of space is conservative if and only if $a_y=b_x$, $a_z=c_x$, and $b_z=c_y$. Is $\boldsymbol{\Phi}(x,y,z)=2xy\mathbf{i}+(3z^3+x^2)\mathbf{j}+9yz^2\mathbf{k}$ conservative? If so, find a function f such that $\nabla f=\boldsymbol{\Phi}$.
24. (a) If $\boldsymbol{\Phi}$ is conservative, prove that $\nabla\times\boldsymbol{\Phi}=\mathbf{0}$.
    (b) Find an f such that
$$\nabla f=\tan^{-1}(yz)\mathbf{i}+\left(1+\frac{xz}{y^2z^2+1}\right)\mathbf{j}+\frac{xy}{y^2z^2+1}\mathbf{k}.$$

Which of the differential forms in Exercises 25–28 are exact? Find antiderivatives for those that are.

25. $(e^y\sin x + xe^y\cos x)\,dx + xe^y\sin x\,dy$.
26. $(y\cos x+\sin z)\,dx + (z\cos y+\sin x)\,dy + (x\cos z+\sin y)\,dz$. (You may wish to study Review Exercise 23 first.)
27. $xe^y\,dx + ye^x\,dy$.
28. $\exp(x^2+y^2)(x\,dx+y\,dy)$.

Solve the differential equations in Exercises 29–32.

29. $y\cos x + 2xe^y + (\sin x + x^2e^y+2)\dfrac{dy}{dx}=0$, $y(\pi/2)=0$.
30. $3x^2-2y+e^xe^y+(e^xe^y-2x+4)\dfrac{dy}{dx}=0$, $y(0)=0$.
31. $2xy+(x^2+1)\dfrac{dy}{dx}=1$, $y(1)=1$.
32. $2xy-2x+(x^2+1)\dfrac{dy}{dx}=0$.

Test each of the equations in Exercises 33–36 for exactness and solve the exact ones.

33. $xy-(x^2+3y^2)\dfrac{dy}{dx}=0$.
34. $\sin x\sin y - xe^y + (e^y+\cos x\cos y)\dfrac{dy}{dx}=0$.
35. $5x^3y^4-2y+(3x^2y^5+x)\dfrac{dy}{dx}=0$.
36. $9x^2+y=(4y-x)\dfrac{dy}{dx}+1$.

37. If D is a region in the plane with boundary curve C traversed counterclockwise, express the following three integrals in terms of the area of D:
(a) $\int_C x\,dy$, (b) $\int_C y\,dx$, (c) $\int_C x\,dx$.
38. Use Green's theorem to calculate the line integral $\int_C (x^3+y^3)\,dy - (x^3+y)\,dx$, where C is the circle $x^2+y^2=1$ traversed counterclockwise.

Calculate the curl and the divergence of the vector fields in Exercises 39–42.

39. $\mathbf{\Phi}(x,y,z) = x\mathbf{i} + [y/(x+z)]\mathbf{j} - z\mathbf{k}$.
40. $\mathbf{\Phi}(x,y,z) = 2xe^y\mathbf{i} - y^2e^z\mathbf{j} + ze^{x-y}\mathbf{k}$.
41. $\mathbf{\Phi}(x,y,z) = \mathbf{r}\times(x\mathbf{i} - y\mathbf{j} - z\mathbf{k})$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
42. $\mathbf{\Phi} = \nabla\times\mathbf{F}$, where $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \cos(yz)\mathbf{j} - \sin(xy)\mathbf{k}$.
43. (a) Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + zx^3y^2\mathbf{k}$. Calculate $\nabla\times\mathbf{F}$ and $\nabla\cdot\mathbf{F}$.
(b) Evaluate $\int\int_{S_1}(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA$ where S_1 is the surface $x^2+y^2+z^2=1$, $z\leqslant 0$.
(c) Evaluate $\int\int_{S_2}\mathbf{F}\cdot\mathbf{n}\,dA$ where S_2 is the surface of the unit cube in the first octant.
44. (a) Let $\mathbf{F} = (x^2+y-4)\mathbf{i} + 3xy\mathbf{j} + (2xz+z^2)\mathbf{k}$. Calculate the divergence and curl of $\mathbf{F}$.
(b) Find the flux of the curl of $\mathbf{F}$ across the surface $x^2+y^2+z^2=16$, $z\geqslant 0$.
(c) Find the flux of $\mathbf{F}$ across the surface of the unit cube in the first quadrant.
45. Express as a surface integral the work done by a force field $\mathbf{F}$ going around a closed curve C in space.
46. Suppose that $\operatorname{div}\mathbf{F} > 0$ inside the unit ball $x^2+y^2+z^2\leqslant 1$. Show that $\mathbf{F}$ cannot be everywhere tangent to the surface of the sphere. Give a physical interpretation of this result.
47. Calculate the surface integral $\int\int_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA$, where S is the hemisphere $x^2+y^2+z^2=1$, $z\geqslant 0$ and $\mathbf{F} = x^3\mathbf{i} - y^3\mathbf{j}$.
48. Calculate the integral of the vector field in Exercise 47 over the hemisphere $x^2+y^2+z^2=1$, $z\leqslant 0$.
49. Calculate the integral $\int\int_S\mathbf{F}\cdot\mathbf{n}\,dA$, where S is the surface of the half ball $x^2+y^2+z^2\leqslant 1$, $z\geqslant 0$, and $\mathbf{F} = (x+3y^5)\mathbf{i} + (y+10xz)\mathbf{j} + (z-xy)\mathbf{k}$.
50. Find $\int\int_S(\nabla\times\mathbf{\Phi})\cdot\mathbf{n}\,dA$, where S is the ellipsoid $x^2+y^2+2z^2=10$ and $\mathbf{\Phi} = \sin xy\mathbf{i} + e^x\mathbf{j} - yz\mathbf{k}$.

For a region W in space with boundary ∂W, unit outward normal $\mathbf{n}$ and functions f and g defined on W and ∂W, prove *Green's identities* in Exercises 51 and 52, where $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ is the Laplacian of f.

51. $$\int\int_{\partial W} f(\nabla g)\cdot\mathbf{n}\,dA = \int\int\int_W (f\nabla^2 g + \nabla f\cdot\nabla g)\,dx\,dy\,dz.$$
52. $$\int\int_{\partial W}(f\nabla g - g\nabla f)\cdot\mathbf{n}\,dA = \int\int\int_W (f\nabla^2 g - g\nabla^2 f)\,dx\,dy\,dz.$$
53. Show that $\operatorname{div}\mathbf{\Phi}$ at a point P_0 in space is the "flux of $\mathbf{\Phi}$ per unit volume" at P_0.
54. In Section 18.4, we gave an example of a region to which Green's theorem as stated did not apply, but which could be cut up into smaller regions to which it did apply. In this way, Green's theorem was extended. Give an example of a similar procedure for Stokes' theorem.
55. Surface integrals apply to the study of heat flow. Let $T(x,y,z)$ be the temperature at a point (x,y,z) in W where W is some region in space and T is a function with continuous partial derivatives. Then
$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$
represents the temperature gradient, and heat "flows" with the vector field $-k\nabla T = \mathbf{F}$, where k is the positive constant. Therefore $\int\int_S\mathbf{F}\cdot\mathbf{n}\,dA$ is the total rate of heat flow or flux across the surface S. ($\mathbf{n}$ is the unit outward normal.)

Suppose a temperature function is given as $T(x,y,z) = x^2+y^2+z^2$, and let S be the unit sphere $x^2+y^2+z^2=1$. Find the heat flux across the surface S if $k=1$.

★56. (a) Express conservation of thermal energy by means of the statement that for any volume W in space
$$\frac{d}{dt}\int\int\int_W e\,dx\,dy\,dz = -\int\int_{\partial W}\mathbf{F}\cdot\mathbf{n}\,dA,$$
where $\mathbf{F} = -k\nabla T$, as in Exercise 55 and $e = c\rho_0 T$, where c is the specific heat (a constant) and ρ_0 is the mass density (another constant). Use the divergence theorem to show that this statement of conservation of energy is equivalent to the statement
$$\frac{\partial T}{\partial t} = \frac{k}{c\rho_0}\nabla^2 T \qquad \text{(heat equation)},$$
where $\nabla^2 T = \operatorname{div}\operatorname{grad} T = \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2$ is the *Laplacian* of T.
(b) Make up an integral statement of conservation of mass for fluids that is equivalent to the *continuity equation*
$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{V}) = 0,$$
where ρ is the mass density of a fluid and $\mathbf{V}$ is the fluid's velocity field.

★57. (a) Let $\mathbf{\Phi}$ be a vector field in space. Follow the pattern of Exercise 31, Section 18.4, replacing Green's theorem by Stokes' theorem to show that $\mathbf{\Phi} = \nabla f$ for some f (that is, $\mathbf{\Phi}$ is conservative) if and only if $\nabla\times\mathbf{\Phi} = \mathbf{0}$.
(b) Is $\mathbf{F} = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ a gradient? If so, find f.

★58. (a) Use Green's theorem to find a formula for the area of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3).
(b) Use Green's theorem to find a formula for the area of the n-sided polygon whose consecutive vertices are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

★59. Let f be a function on the region W in space such that: (i) $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$ everywhere on W and (ii) ∇f is tangent to the boundary ∂W at each point of ∂W. Use the identity in Exercise 51 to prove that f is constant.

★60. Show that the result in Exercise 59 is true if the condition (ii) is replaced by (ii′) f is constant on ∂W.

★61. Considering a closed curve in space as the boundary of two different surfaces, discuss the relation between:
(a) the divergence theorem;
(b) Stokes' theorem;
(c) the identity $\nabla \cdot (\nabla \times \mathbf{\Phi}) = 0$.